Faster Implied Volatilities via the Implicit Function Theorem

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Faster Implied Volatilities via the Implicit Function Theorem

Abstract

We present a faster, more accurate technique for estimating implied volatility using the standard partial derivatives of the Black-Scholes options-pricing formula. Beside Newton-Raphson and slower approximation methods, this technique is the first to provide an error tolerance, which is essential for practical application. All existing non-iterative approximation methods do not provide error tolerances and have the potential for large errors.
1. Introduction

Since Black and Scholes (1973) unveiled their option pricing formula, practitioners and theoreticians have been estimating the volatility implied by option prices. The impact has been so profound that investment professionals and academics often ignore option prices, focusing instead upon the implied volatility.\(^1\)

Whatever the impact of implied volatility, it must be calculated. The Black-Scholes options-pricing formula cannot be inverted, so an approximation must be used. We present a faster, more accurate technique for estimating implied volatility using the standard partial derivatives of the Black-Scholes options-pricing formula. This is the first non-iterative technique to provide an error tolerance, which is critical since all existing non-iterative approximation methods have the potential for large errors.

Existing approximation techniques can be classified as follows. First, numerically iterative techniques, such as Newton-Raphson, are computationally intensive, yet provide an error tolerance and can be made to converge.\(^2\) Second, quasi-iterative techniques, such as those proposed by Chance (1996) and Chambers and Nawalkha (2001) use a second-order Taylor expansion of the call option function in volatility and strike to create a quadratic in implied volatility that is solved using the quadratic formula. Chambers and Nawalkha restrict Chance’s Taylor expansion to be only in volatility, improving its accuracy. These methods are quasi-iterative since they require an \textit{a priori} estimate of the implied volatility to serve as a starting point. Neither method provides an error tolerance.

Third, Brenner and Subrahmanyam (1988), Corrado and Miller (1996), and Hallerbach (2004) present non-iterative, closed form approximations. Brenner and Subrahmanyam present a formula for calculating the implied volatility of at-the-money forward options using a linear approximation of the cumulative normal distribution function. Corrado and Miller use a quadratic approximation of the cumulative normal distribution function to create a second-order polynomial in the implied volatility that they solve using the quadratic formula. There is no at-the-money restriction. Hallerbach solves a quadratic and uses approximation techniques to derive a more precise implied volatility estimate than that of Corrado-Miller. None of the non-iterative methods provide an error tolerance.

We take a different approach in this paper. Rather than approximating the option price function: \(^3\)

\[ C = f(\sigma) \]

\(^1\) For instance, see Chan, Chen, and Lung (2004) where implied volatility is related to equity returns.
\(^3\) We focus on European call options, but the analysis can be extended to European put options.
We consider $\sigma(C)$ as a function of $C$. Hence,

$$C = f(\sigma(C))$$

Using the implicit function theorem, we calculate the partial derivatives of $\sigma(C)$ to Taylor expand $\sigma(C)$.

$$\sigma_1 = \sigma_0 + \frac{\partial \sigma}{\partial C}(C_1 - C_0) + \frac{1}{2} \frac{\partial^2 \sigma}{\partial C^2}(C_1 - C_0)^2 + \text{higher order terms}$$

Our procedure is quasi-iterative, in the spirit of Chance (1996) and Chambers-Nawalkha (2001). Similar to Chambers-Nawalkha, we restrict our Taylor expansion to be in $C$, for greater accuracy. Importantly, a criterion exists that allows the estimate to be used only when a pre-specified tolerance is met. Since the error can be quite large, an error tolerance is essential for any estimate to have practical application. No previous, non-iterative method provides an error tolerance level.

2. Method

We consider the Black-Scholes options-pricing formula for a European call option without dividends:

$$C = SN(d_1) - Ke^{-rt}N(d_2)$$

where

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)t}{\sigma \sqrt{t}}$$
$$d_2 = d_1 - \sigma \sqrt{t}$$

with

$S =$ stock price
$K =$ strike price
$r =$ continuously compounded interest rate
$t =$ time to option maturity (tenor)
$N(\cdot) =$ cumulative normal distribution function

To employ the implicit function theorem (see Hubbard and Hubbard 2002, chap. 2), the formula is rewritten as:

$$u(C, \sigma(C)) = SN(d_1) - Ke^{-rt}N(d_2) - C = 0$$

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The method can be extended to European put options. For European call options, the inclusion of dividends is straightforward. The analysis can also be extended to include European options on futures and foreign exchange. For American options, the implicit function technique could be applied to the formula proposed by Roll (1997), Geske (1979, 1981), and Whaley (1981). Hull (1997, p. 259) offers a summary.
A Taylor expansion in two variables \((C, S)\) has poor convergence for out-of-the-money options, so we restrict ourselves to an expansion in \(C\). \(C > 0\) implies that 
\[\frac{\partial u}{\partial \sigma} = \text{vega} > 0.\]
Hence, an implicit function, \(\sigma(C)\), exists for some region around any positive \(C_0\). The derivatives of \(\sigma(C)\) are calculated with respect to \(C\) by using the chain rule. These derivatives allow us to Taylor expand \(\sigma(C)\) to any order.\(^5\)

\[
\sigma_1 = \sigma_0 + \frac{\partial \sigma}{\partial C}(C_1 - C_0) + \frac{1}{2} \frac{\partial^2 \sigma}{\partial C^2}(C_1 - C_0)^2 + \text{higher order terms}
\]

Stock and option prices typically move simultaneously. By only expanding around \(C_0\), we appear to limit the usefulness of our approximation technique. Let us consider an application to see how the technique could be used.

Most real-time volatility calculators are market monitors. That is, each time the stock price or the option price changes, the implied volatility is updated. Except for the first calculation, the previous implied volatility is available to calculate the next implied volatility.

Suppose that the stock price moves from \(S_0\) to \(S_1\). We use the new stock price, \(S_1\), and the previously calculated implied volatility, \(\sigma_0\), to calculate a Black-Scholes option value, \(C(S_1, \sigma_0)\), and its derivatives. We then calculate the partial derivatives of \(\sigma(C)\) evaluated at \((S_1, C(S_1, \sigma_0))\) and calculate our estimate for volatility using the Taylor expansion around \(C(S_1, \sigma_0)\) extrapolated to the observed price of the option, \(C_1\).\(^6\) Since we need a starting implied volatility for our calculations, our technique is quasi-iterative, in the spirit of Chance (1996) and Chambers and Nawalkha (2001).

3. Results

Estimated volatilities based upon a fifth-order expansion of \(\sigma(C)\) are presented in Table 1. We have found that a fifth-order expansion provides, in most cases, fairly accurate estimates for a broad range of moneyness and maturities. The option is a 3-month, at-the-money call with implied volatility equal to 30% and interest rates equal to zero.\(^7\) The calculated implied volatilities in the table use 30% as the seed. The column headings represent the “true” implied volatility, while the rows represent the stock price.

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\(^5\) The derivatives appear in Appendix to the fifth order. The evaluation of the derivatives for a given \((S_0, C_0, r_0, t_0)\) only requires the standard call option derivatives calculated by most software packages.

\(^6\) Similarly, the current interest rate, \(r_1\), is used to calculate the Taylor expansion.

\(^7\) We used \(r=0\), so that the at-the-money forward is equal to the current spot price. The estimation technique is valid for any interest rate.
Table 1  
Results for 5th order expansion of $\sigma(C)$. $K = 100; r = 0; t = 0.25; \sigma_0 = 30\%$

<table>
<thead>
<tr>
<th></th>
<th>15.00%</th>
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<td>22.88%</td>
<td>23.97%</td>
<td>26.09%</td>
<td>28.00%</td>
<td>30.00%</td>
<td>32.01%</td>
<td>34.96%</td>
<td>93.49%</td>
<td>24558.45%</td>
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For stock prices between 70 and 140, the estimated implied volatilities are within 0.1% for the range of 28 to 32% true implied volatilities. For most real-time calculations, this is well within the range of implied volatility for a single day. For stock prices between 85 and 120, the range extends to 23 to 37%. The option premium when the stock price is 70 is 0.04, a level low enough to expect that the 5th order approximation would break down. When the stock price is 140, the option premium over intrinsic is 0.07, a level at which we would expect problems with the approximation.

While the results of the 5th order expansion are accurate over a large range, they do eventually become highly inaccurate. (For the case where $S = 140$ and $\sigma_1 = 45\%$, the estimated volatility is 24,558%!) To avoid these errors, an intuitive criterion is available to determine whether the extrapolation is within a particular tolerance. Suppose we know $\sigma_0$. The stock price moves to $S_1$ and we calculate $C(\sigma_0, S_1)$. For a call option price, $C_1$, the 5th order estimate is within 0.1% tolerance if:

$$\left| \frac{C_1 - C(\sigma_0, S_1)}{C(\sigma_0, S_1) - \max(S - e^{-rt}K, 0)} \right| < 0.64$$

when $\sigma_0 < 50\%$ and $t < 3$ years

or

$$\left| \frac{C_1 - C(\sigma_0, S_1)}{C(\sigma_0, S_1) - \max(S - e^{-rt}K, 0)} \right| < 0.40$$

when $\sigma_0 < 100\%$ and $t < 3$ years

or

$$\left| \frac{C_1 - C(\sigma_0, S_1)}{C(\sigma_0, S_1) - \max(S - e^{-rt}K, 0)} \right| < 0.10$$

when $\sigma_0 < 150\%$ and $t < 3$ years

For an initial volatility less than 50% and a tenor less than 3 years, if the absolute value of the percentage change of the call premium is less than 64%, the extrapolation is within 0.1% of the true volatility. The call premium in the denominator is adjusted by

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8 Our approximation technique can be used for $\sigma_0 > 150\%$ by calculating the appropriate criterion cut-off levels. While the technique may be used for $t > 3$ years at a lower cut-off level, we restrict ourselves to $t < 3$ years since most standard options in the United States have tenors less than 3 years.
subtracting the intrinsic value of the option adjusted for the time value of the strike (i.e., the “discounted intrinsic” or max(S – e^{rT}K, 0)).

Cut-off levels can be computed for other tolerances (e.g., 0.01%) and for higher or lower order polynomial expansions. By computing cut-off levels for the 1st through nth order polynomials, an implied volatility calculator need only use the smallest degree polynomial required to remain within tolerance.

If the cut-off level is violated, the implied volatility calculator reverts to Newton-Raphson or some other method. The implied volatility obtained from Newton-Raphson then is used as the starting point for subsequent extrapolations of σ.

These cut-off levels were determined by examining all stock prices and volatilities where the option premium is 0.001 above discounted intrinsic. The strike used is 100, so the excluded options have premiums less than 0.001% of the strike. The stock price increment is 1. The volatility increment is 0.1% for σ0 <= 30%, 0.2% for σ0 <= 75%, and 1% for σ0 > 75%. Varying interest rates has no effect on the results. A more detailed look at the cut-off levels appears in Figure 1.

![Figure 1](image-url)

**Cut-off levels for 0.1% tolerance vs. volatility for various option tenors**

The cut-off levels shown in Figure 1 appear to have a well-behaved pattern that might be determined by using some results of Carr (2000). Carr has calculated the general solutions for all nth order derivatives of the Black-Scholes options-pricing formula with respect to stock price, volatility, interest rate, time, and dividend yield. A functional form for the error term might be found by using Carr’s nth order derivative equations to estimate the error term of the Taylor expansion.

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9 For a 5th order approximation, the cut-off level for 0.01% tolerance is 0.45 for σ0 < 50% and t < 3 years, 0.2 for σ0 < 100% and t < 3 years, and 0.07 for σ0 < 150% and t < 3 years. For a 1st order approximation (linear), the cut-off level for a 0.1% tolerance is 0.13 for σ0 < 50% and t < 3 years.
Carr (2000) also shows that the $n^{\text{th}}$ order Taylor expansion in volatility converges as $n$ increases only in a radius $\sigma_0/\sqrt{2}$ around $\sigma_0$. We speculate that a similar radius exists around $C_0$ for the Taylor expansion of the implicit function, $\sigma(C)$, and that the tolerance levels are strictly within this radius. The existence of a finite convergence radius is likely why a global approximation for implied volatility has not been found.

4. Comparison to previous results

The 5th order approximation of $\sigma(C)$ should compare well to non-iterative approximations, such as those of Corrado-Miller (1996) and Hallerbach (2004), for three reasons. For the 5th order approximation, the stock price is fixed while the non-iterative approximations are “global” approximations that are good for any stock price and do not extrapolate from an initial volatility. Also, the 5th order approximation has an error tolerance, assuring against large errors in the estimate.

We have compared the 5th order approximation to the Hallerbach (2004) approximation and find that the 5th order approximation is considerably more accurate for a wide range of maturities and moneyness. For applications for which a quasi-iterative approximation is appropriate, the 5th order approximation is preferred to the Corrado-Miller (1996) and Hallerbach approximations. However, Hallerbach’s estimates are quite accurate for a wide range of volatilities, moneyness, and maturities. His estimate is a good potential “seed” for our quasi-iterative technique.

We now compare the 5th order estimate to that of Chance (1996), and Chambers and Nawalkha (2001). Chambers and Nawalkha convert Chance’s two-variable estimator into a quasi-iterative, univariate estimator, like the 5th order approximation. No error tolerance for their estimate exists. Table 2 shows the comparison of the Chance-Chambers-Nawalkha estimate to the 5th order estimate. Table 1 illustrates that the 5th order approximation is highly accurate for options that are near-the-money forward, hence we focus on options that are in and out-of-the-money forward. Case 1 is an out-of-the-money, extremely short-dated option. Case 2 is a deep out-of-the-money, medium maturity option. Case 3 is an in-the-money forward, long-dated option.

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10 For a one-week option, a 2% difference between the strike and stock price is fairly out-of-the-money.
11 While Hallerbach (2004) lists his results solely based upon the aggregate volatility, $\sigma\sqrt{t}$, we cannot since we have taken derivatives with respect to $\sigma$.  

7
Table 2

Comparison to Chance-Chambers-Nawalkha. (K = 100; \( \sigma_0 = 30\% \); r = 5%)

<table>
<thead>
<tr>
<th>Stock Price = 98</th>
<th>Stock Price = 80</th>
<th>Stock Price = 100</th>
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</thead>
<tbody>
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<td>40%</td>
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</tr>
<tr>
<td>45%</td>
<td>44.768%</td>
<td>45.092%</td>
</tr>
</tbody>
</table>

As expected the 5th order approximation outperforms the Chambers-Nawalkha approximation; however, for many volatility values, the difference is slight. The 5th order approximation shows the most improvement when extrapolating downward.

5. Conclusion

The implicit function theorem and Taylor’s theorem are powerful tools for estimating the implied volatility function. The estimation of \( \sigma(C) \) is accurate for a wide range of tenors, volatilities, and interest rates. A criterion exists so that the estimator is used only when a tolerance is satisfied.

The 5th order Taylor expansion of \( \sigma(C) \) should be of particular use for real-time implied volatility calculators. If greater precision is needed, a higher order Taylor expansion can be calculated. The simple criterion of the call option premium moving by no more than a fixed percentage of discounted intrinsic is intriguing and points to the potential of using this technique to gain a greater insight into the behavior of the implied volatility function.
Appendix

We present the partial derivatives of $\sigma(C)$. To calculate the first partial derivative with respect to $C$, we employ the chain rule on $u(C, \sigma(C))$:

$$\frac{\partial u}{\partial C} + \frac{\partial u}{\partial \sigma} \cdot \frac{\partial \sigma}{\partial C} = 0$$

Since $\frac{\partial u}{\partial C} = -1$ and $\frac{\partial u}{\partial \sigma} = \text{vega}$,

$$\frac{\partial \sigma}{\partial C} = \frac{1}{\text{vega}}$$

The higher-order derivatives follow. The cross-partials of $u(\cdot)$ with respect to $C$ and $\sigma$ as well as the second-order and higher partials with respect to $C$ are zero:

$$\frac{\partial^2 \sigma}{\partial C^2} = -(1/\partial u/\partial \sigma) \cdot \partial^2 u/\partial \sigma^2 \cdot (\partial \sigma/\partial C)^2$$

$$\frac{\partial^3 \sigma}{\partial C^3} = -(1/\partial u/\partial \sigma) \cdot (\partial^3 u/\partial \sigma^3) \cdot (\partial \sigma/\partial C)^3 + 3 \partial^2 u/\partial \sigma^2 \cdot \partial \sigma/\partial C \cdot \partial^3 \sigma/\partial C^2$$

$$\frac{\partial^4 \sigma}{\partial C^4} = -(1/\partial u/\partial \sigma) \cdot (\partial^4 u/\partial \sigma^4) \cdot (\partial \sigma/\partial C)^4 + 6 \partial^3 u/\partial \sigma^3 \cdot (\partial \sigma/\partial C)^3 \cdot \partial^2 \sigma/\partial C^2$$

$$+ 3 \partial^2 u/\partial \sigma^2 \cdot (\partial^3 \sigma/\partial C^2) + 4 \partial^2 u/\partial \sigma^2 \cdot \partial \sigma/\partial C \cdot \partial^3 \sigma/\partial C^2$$

$$\frac{\partial^5 \sigma}{\partial C^5} = -(1/\partial u/\partial \sigma) \cdot (\partial^5 u/\partial \sigma^5) \cdot (\partial \sigma/\partial C)^5 + 10 \partial^4 u/\partial \sigma^4 \cdot (\partial \sigma/\partial C)^4 \cdot \partial^3 \sigma/\partial C^2$$

$$+ 15 \partial^3 u/\partial \sigma^3 \cdot \partial \sigma/\partial C \cdot (\partial^2 \sigma/\partial C^2) \cdot (\partial \sigma/\partial C)^3 + 10 \partial^3 u/\partial \sigma^3 \cdot \partial \sigma/\partial C \cdot \partial^2 \sigma/\partial C^2$$

$$+ 5 \partial^2 u/\partial \sigma^2 \cdot \partial \sigma/\partial C \cdot \partial \sigma/\partial C \cdot \partial^2 \sigma/\partial C^2$$

where\(^{12}\)

$$\frac{\partial^2 u}{\partial \sigma^2} = -\text{vega} \cdot (d_1 d_2 / \sigma)$$

$$\frac{\partial^3 u}{\partial \sigma^3} = (d_1 d_2 - d_2 d_1 - d_2 d_1 - d_1 d_2) / \sigma^2 \cdot \text{vega}$$

$$\frac{\partial^4 u}{\partial \sigma^4} = ((d_1 d_2 - 2) / \sigma) \cdot \frac{\partial^3 u}{\partial \sigma^3} \cdot ((d_1 + d_2 + 4d_1 d_2 - 2d_1 d_2 - 2d_1 d_2) / \sigma^3 \cdot \text{vega}$$

$$\frac{\partial^5 u}{\partial \sigma^5} = ((d_1 d_2 - 3) / \sigma) \cdot \frac{\partial^4 u}{\partial \sigma^4} \cdot ((d_1 + d_1 + d_2 + d_1) / \sigma) \cdot \text{vega}$$

$$+ ((d_1 d_2 - 2) / \sigma) \cdot (d_1 + d_2 + 4d_1 d_2 - 2d_1 d_2 - 2d_1 d_2) / \sigma^3 \cdot \text{vega}$$

$$+ (2d_1 + 2d_2 + 12d_1 d_2 - 4d_1 d_2 - 4d_1 d_2) / \sigma^4 \cdot \text{vega}$$

Note that all partials of $u(\cdot)$ with respect to $\sigma$ are a function of $d_1$, $d_2$, $\sigma$, $t$, and $\text{vega}$. Hence, the existing data model of most option software packages need not be changed to accommodate these calculations.

End Appendix

\(^{12}\) Note that $\partial d_1 / \partial \sigma = -(d_1 / \sigma)$ and $\partial d_2 / \partial \sigma = -(d_2 / \sigma)$ or, alternatively,

$$\partial d_1 / \partial \sigma = \sqrt{t} - (d_1 / \sigma)$$

and $\partial^n d_1 / \partial \sigma^n = (-1)^n (n! / 2) (\sigma^{n-1}) - (-1)^n (n! / 2) (\sigma^{n-1})$ for $n > 1$. \textit{End Appendix}
References


