A bracket polynomial for graphs. II.
Links, Euler circuits and marked graphs

Lorenzo Traldi
Lafayette College
Easton, Pennsylvania 18042

Abstract
Let \( D \) be an oriented classical or virtual link diagram with directed universe \( \vec{U} \). Let \( C \) denote a set of directed Euler circuits, one in each connected component of \( U \). There is then an associated looped interlacement graph \( \mathcal{L}(D, C) \) whose construction involves very little geometric information about the way \( D \) is drawn in the plane; consequently \( \mathcal{L}(D, C) \) is different from other combinatorial structures associated with classical link diagrams, like the checkerboard graph, which can be difficult to extend to arbitrary virtual links. \( \mathcal{L}(D, C) \) is determined by three things: the structure of \( \vec{U} \) as a 2-in, 2-out digraph, the distinction between crossings that make a positive contribution to the writhe and those that make a negative contribution, and the relationship between \( C \) and the directed circuits in \( \vec{U} \) arising from the link components; this relationship is indicated by marking the vertices where \( C \) does not follow the incident link component(s).

We introduce a bracket polynomial for arbitrary marked graphs, defined using either a formula involving matrix nullities or a recursion involving the local complement and pivot operations; the marked-graph bracket of \( \mathcal{L}(D, C) \) is the same as the Kauffman bracket of \( D \). This provides a unified combinatorial description of the Jones polynomial that applies seamlessly to both classical and non-classical virtual links.

Keywords. graph, virtual link, Kauffman bracket, Jones polynomial, Euler circuit, circuit partition, trip matrix, interlacement, Reidemeister move

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1 Introduction
A classical link \( L \) consists of finitely many, pairwise disjoint, piecewise smooth closed curves in \( \mathbb{R}^3 \) or \( S^3 \); each individual closed curve is a component of \( L \). A regular diagram \( D \) of \( L \) is obtained from a regular projection in the plane – that is, a projection whose only singularities are double points called crossings – by specifying the over-and-under-crossing arcs at each crossing. The projection
itself is a 4-regular plane graph \( U \), the \textit{universe} of \( D \). Saying that \( U \) is a \textit{plane graph} means that it is given with a specific embedding in \( \mathbb{R}^2 \); when the embedding is forgotten \( U \) becomes an \textit{abstract graph}. We presume \( L \) is given with orientations of its components; these orientations make \( U \) into a 2-in, 2-out digraph \( \vec{U} \). Combinatorial structures associated with \( D \) incorporate geometric information about its embedding in the plane in various ways – for instance the classical checkerboard graph is defined using the connected components of \( \mathbb{R}^2 - U \), a rigid graph specifies for each vertex of \( U \) the cyclic order of the incident edges, a signed Gauss code specifies the cyclic order of the crossings on each link component along with underpassing-overpassing information at each crossing, and an atom includes an embedding of a graph on a surface.

Most combinatorial structures associated to classical link diagrams must be adjusted when we try to extend them to \textit{virtual link diagrams} \cite{21}. Each crossing in a virtual link diagram has three possible configurations. In addition to the two classical configurations with different choices of under- and over-crossing arcs, there is the virtual configuration, in which two arcs of the diagram simply cross each other without intersecting. The virtual crossings are “not really there” \cite{21}, in the same way that a plane representation of a graph may contain apparent edge-intersections that are “not really there.” The universe of a virtual link diagram is an abstract 4-regular graph with vertices corresponding only to the classical crossings; the virtual diagram gives a regular immersion of the graph in \( \mathbb{R}^2 \), not an embedding as in the classical case. The virtual crossings naturally suggest an embedding of the universe on a closed, orientable surface \( S \) of some genus, and the diagram can then be thought of as representing a link in the thickened surface \( S \times \mathbb{R} \). Consequently one way to extend notions involving the plane geometry of classical link diagrams is to use the geometry of the surface \( S \); for instance the classical checkerboard graphs extend in this manner to atoms associated with virtual diagrams. (By the way we should make it clear that we use the terms \textit{link}, \textit{virtual link} and \textit{classical or virtual link} interchangeably; when we want to be restrictive we specify that a link is classical or non-classical.)

The Kauffman bracket and Jones polynomial of a classical link \( L \) may be described in a variety of ways using combinatorial structures associated with regular diagrams of \( L \); Kauffman’s basic description follows. Each crossing of a regular diagram \( D \) has two \textit{smoothings}, one denoted \( A \) and the other denoted \( B \); distinguishing between them is another example of the use of geometric information mentioned in the first paragraph above, as it involves the orientation of the plane in which \( D \) is drawn. If \( D \) has \( n \) crossings then it has \( 2^n \) \textit{states}, obtained by applying either the \( A \) or the \( B \) smoothing at each crossing. Given a state \( S \) let \( a(S) \) denote the number of \( A \) smoothings in \( S \), \( b(S) = n - a(S) \) the number of \( B \) smoothings in \( S \), and \( c(S) \) the number of simple closed curves in \( S \), including any crossing-free components that might appear in \( D \). (These simple closed curves are called “loops” in \cite{21}, but we reserve this word for graph-theoretic use.) Then the \textit{Kauffman bracket polynomial} of \( D \) is a sum

\[ 
\]
Almost immediately after the Kauffman bracket was introduced, it was related to the Tutte polynomial through the checkerboard graph construction \cite{12}. A classical link diagram $D$ has an alternating orientation on its universe $U$, i.e., the edges of $U$ may be directed in such a way that when one traverses any component of the link, one encounters forward- and backward-directed edges alternately. Such an alternating orientation arises naturally from the cycles that bound the complementary regions of $D$, and is associated with the relationship between Kauffman states of $D$ and spanning forests in the checkerboard graph that underlies the Jones-Tutte connection. This connection between spanning forests and Kauffman states is also part of a combinatorial discussion of the Alexander polynomial \cite{20}. By the way, the relationship between Eulerian circuits and spanning trees in regular directed graphs, known as the BEST theorem, appeared in the combinatorial literature long before the Jones polynomial was introduced \cite{13,40}; the special relationship between circuit partitions in a 4-regular plane graph and the Tutte polynomial of an associated checkerboard graph also appeared first outside knot theory \cite{10,32}.

The Kauffman bracket of a virtual link diagram is described in almost exactly the same way as that of a classical link diagram; the only difference is that we cannot say that $c(S)$ denotes the number of simple closed curves in $S$, as the closed curves need not be simple. Much of the Jones-Tutte machinery extends directly to virtual link diagrams which possess alternating orientations or (equivalently) orientable atoms; see \cite{18} and also Chapter IV of \cite{28}. If the Tutte polynomial is augmented with topological information to give an invariant of graphs embedded on surfaces as in \cite{5,6}, then the relationship between links in thickened surfaces and virtual links leads to a general relationship between this topological Tutte polynomial and the Jones polynomial \cite{9,10}.

In \cite{44} Zulli and the present author proposed a different, purely combinatorial approach to the Jones polynomials and Kauffman brackets of classical and
virtual knots (one-component links) that uses surprisingly little geometric information about knot diagrams, and does not involve either the ordinary Tutte polynomial or the topological Tutte polynomial. If $D$ is a regular diagram of a classical or virtual knot $K$ then the universe $U$ of $D$ has an interlacement graph defined with respect to the Euler circuit obtained directly from $K$, and the Kauffman bracket $[D]$ can be obtained from this interlacement graph without using complementary regions, ordering the crossings, specifying over- and under-crossing arcs, ordering the edges incident on a vertex, distinguishing between smoothings, counting closed curves, or considering closed surfaces of any genus. All that is required is to record whether each crossing’s contribution to the writhe is +1 or -1; this is done by attaching a loop to each vertex that corresponds to a crossing whose writhe contribution is -1. The resulting looped interlacement graph is denoted $L(D)$. $L(D)$ is obtained by considering $U$ as an abstract graph rather than an embedded one, so any virtual crossings are really not there (rather than merely “not really there”) as far as $L(D)$ is concerned.

The analysis of [44] describes the Jones polynomial of a classical or virtual knot as the result of the composition of three functions. The first function is the looped interlacement graph construction. The second function associates a 3-variable graph bracket polynomial to an arbitrary graph, using either a recursion involving the local complement and pivot operations that have appeared in the theory of circle graphs and interlace polynomials [1, 2, 3, 7, 8, 25] or a formula involving matrix nullities over $GF(2)$. Zulli [46] proved that this formula, when applied to looped interlacement graphs, correctly assesses the number of closed curves in each Kauffman state. The third function modifies the 3-variable graph bracket polynomial so that the resulting graph Jones polynomial is invariant under appropriate graph Reidemeister moves.

\[
D \mapsto L(D) \mapsto [L(D)] \mapsto V_{L(D)}(t)
\]

(virtual) looped interlacement graph using 3-variable graph bracket polynomial graph

The purpose of the present paper is to present an extension of this description to the Jones polynomials of classical and virtual links. Each of the three steps must be modified to accommodate the extension.

\[
D \mapsto L(D, C) \mapsto [L(D, C)] \mapsto V_{L(D, C)}(t)
\]

(virtual) looped interlacement graph using 3-variable marked-graph polynomial

An Euler system $C$
Figure 2: Marked crossings indicate the transitions of a directed Euler system.

The first step is again the construction of a looped interlacement graph. An interlacement graph of a 4-regular graph or a 2-in, 2-out digraph is constructed by using a specific Euler circuit, or if the original graph is not connected a specific Euler system containing an Euler circuit for each connected component. A classical or virtual knot diagram has an Euler circuit arising naturally from the diagrammed knot, but in general there is no canonical choice of an Euler system in a universe $U$ associated to a diagram $D$ representing a multi-component link. Consequently our discussion of this step in Section 2 involves two complications that did not affect [44]: $U$ need not be connected, and we must allow a variety of Euler systems in $U$.

The first of these complications is accommodated using a common graph-theoretic convention, which allows graphs to include free loops; a free loop is not incident on any vertex, but does count as a connected component. If $U$ is a disconnected graph with $c(U) > 1$ connected components, we “record” this fact by inserting $c(U) - 1$ free loops into the interlacement graph $L(D, C)$. This might seem unnatural at first, but it is necessary if we want to recover the Kauffman bracket from $L(D, C)$; without some such convention interlacement cannot detect crossing-free unknotted components, or (more generally) distinguish connected sums from split unions.

The second complication is handled by adopting a convention to identify the Euler system used to produce a looped interlacement graph $L(D, C)$. We presume the link $L$ is given with orientations of its components, and we restrict our attention to directed Euler systems of the directed universe $\vec{U}$. Each crossing of $D$ is then marked to record which of the two possible transitions (edge-pairings) occurs at that crossing in the Euler system $C$ used to produce $L(D, C)$. (Transitions provide a standard combinatorial description of circuits in 4-regular
Figure 3: Marked diagrams and interlacement graphs indicate Euler systems.

graphs and 2-in, 2-out digraphs; see for instance [14, 16, 25]. Crossings at which $C$ follows the incident link component(s) are left unmarked, and crossings at which $C$ is orientation-consistent with the incident link component(s) but does not follow it (or them) are marked with the letter $c$; see Figures 2 and 3. The same convention is used to mark the vertices of $U$ and $L(D,C)$. According to results discussed in [1, 2, 25, 37, 45], different directed Euler systems of $\vec{U}$ are interconnected through transposition, a combinatorial operation. It follows that changing the choice of $C$ affects $L(D,C)$ through an appropriately modified version of the pivot operation, which we call a marked pivot.

As in [44], the second step uses a formula involving matrix nullities over $GF(2)$ to associate a 3-variable bracket polynomial to an arbitrary graph; but now we consider arbitrary graphs with marked vertices. The marked-graph bracket polynomial of a looped interlacement graph $L(D,C)$ is the same as the Kauffman bracket polynomial of $D$; the relationship between the matrix nullity formula and the Kauffman bracket is verified using an extension of a combinatorial equality due to Cohn and Lempel [11]. This equality is presented briefly in Section 4, and more fully in [43]; it incorporates the Cohn-Lempel
equality along with results used in [26, 33, 46].

In Section 5 we define the 3-variable marked-graph bracket polynomial and deduce that $|\mathcal{L}(D,C)| = |D|$. We also prove a very useful result: the bracket is invariant under the marked pivot operation mentioned above. This invariance allows us to develop the theory of the marked-graph bracket using many of the results of [44], even though marked graphs did not appear there; in essence, we often use marked pivots to simply push marked vertices out of the way. For instance, in Section 6 we show that the recursion given in [44] extends to a recursive description of the marked-graph bracket polynomial involving the local complementation and pivot operations. The recursion uses formulas that can only be applied to special configurations in which certain vertices have no marked neighbors; such configurations can always be created using marked pivots.

The third step involves proving that the marked-graph bracket polynomial yields a marked-graph Jones polynomial that is invariant under certain Reidemeister moves on marked graphs; these graph moves generate the “images” of the familiar Reidemeister moves of knot theory under the construction of $\mathcal{L}(D,C)$. We say generate because we do not explicitly exhibit all the marked-graph Reidemeister moves; following an idea of Östlund [35], we only exhibit enough basic moves to generate the rest through composition. As in [44], the fact that $\mathcal{L}(D,C)$ is defined by considering $U$ as an abstract graph rather than a graph drawn on a plane or a surface is reflected in the fact that we can simply ignore the virtual Reidemeister moves of [21].

In sum, our results show that the Kauffman bracket and Jones polynomial can be defined for either of the following abstract (non-embedded) combinatorial structures.

- a 2-in, 2-out digraph $\vec{U}$, given with a partition of its edge-set into directed circuits and a partition of its vertex-set into two subsets
- an undirected graph $G$, given with a partition of its vertex-set into two subsets

It would be interesting to understand how these two types of abstract combinatorial structures are related to other knot-theoretic notions. For instance, orientable and non-orientable atoms have been used for a variety of purposes, e.g., to construct Khovanov homology for virtual knots [29]. Atoms involve graphs embedded on surfaces, so they contain more geometric information than the abstract combinatorial structures. Perhaps some constructions based on atoms do not require all of this geometric information, and can be modified to use one of the abstract combinatorial structures. Manturov has also recently introduced the idea of free knots and links, which are equivalence classes of
4-regular graphs under modified versions of the Reidemeister moves [30, 31]; they are related to the first kind of abstract combinatorial structure mentioned above, with the vertex-partition forgotten. Perhaps the combinatorial theory of circuit partitions in 4-regular graphs will turn out to be useful in defining invariants of these objects.

It would also be interesting to conduct a purely combinatorial investigation of the bracket as an invariant of either of these structures, apart from the special case in which $U$ is the universe of a diagram of an oriented link, the circuit partition of $E(U)$ consists of the link’s components, the partition of $V(U)$ represents the distinction between positive and negative crossings, $G$ is the looped interlacement graph of $U$ with respect to an Euler system $C$, the partition of $V(G)$ represents vestigial information regarding $C$, and free loops represent all but one of the connected components of $U$. In particular, we wonder whether there might be a useful relationship between the bracket polynomial and the interlace polynomials of [1, 2, 3] or the multivariate interlace polynomial of [12]. The same comments apply to the Jones polynomial, which is naturally regarded as a simplification of the bracket, modified so as to be invariant under a certain equivalence relation which is motivated topologically but may be described in purely combinatorial terms.

Before closing this introduction we should thank L. Zulli for many hours spent working together to formulate and make precise the ideas of [14], and also for his comments on early drafts of the current paper. D. P. Ilyutko was kind enough to bring [15] to our attention; the present work developed as we tried to understand the differences and similarities of the two approaches. V. O. Manturov’s good advice improved the paper in several ways. We are also grateful to Lafayette College for its support.

2 Euler circuits and interlacement graphs

In this section we discuss some terminology and results about Eulerian graphs, and then apply these notions to link diagrams. Eulerian graphs have been studied for centuries; it is not surprising that terminology has varied over the years, and some results have been rediscovered. For us a graph may have loops or multiple edges, and each edge has two distinct directions; in a directed graph a preferred direction has been chosen for each edge. The terms adjacent and neighbor refer to non-loop edges only; no vertex is adjacent to itself or a neighbor of itself. In order to provide appropriate notation for the two different directions of a loop, each edge is technically regarded as consisting of two distinct half-edges, one incident on each end-vertex; the preferred direction of a directed edge is expressed by designating one half-edge as initial and the other as terminal. We will almost always abuse notation by leaving it to the reader to split edges into half-edges. A path in a graph is a sequence $v_1, e_1, v_2, e_2, ..., v_{n-1}, e_{n-1}, v_n$ (technically $v_1, h_1, v_2, h_2, v_3, h_3, ..., v_n, h_n, v_{n+1}$) such that for each $i < n$, the vertices $v_i$ and $v_{i+1}$ are incident on
the edge $e_i$; in a directed graph a directed path must respect all edge-directions, and an undirected path might not respect some edge-directions. If $v_1 = v_n$ the path is closed; in this case it is customary but not mandatory to omit $v_n$ from the list. A circuit is a closed path in which no edge appears more than once; digraphs have directed circuits and undirected circuits. An Euler circuit in a connected graph is a circuit in which every edge of the graph appears, and an Euler system in a graph is a set that contains one Euler circuit for each connected component. A connected, undirected graph has an Euler circuit if and only if every vertex is of even degree; a connected, directed graph has a directed Euler circuit if and only if every vertex has equal indegree and outdegree.

We allow graphs to have free loops. These do not interact in any way with the vertices and edges of the graph, and they do not affect anything we would otherwise say about the graph, except for the fact that each free loop is counted as a connected component. In particular, a free loop is not a kind of loop; indeed if the terminology were not standard we would call them “empty components.” For example, if an undirected graph $G$ consists of a 4-cycle and two free loops then $G$ is 2-regular, and if $e_1, e_2$ are two edges in $G$ that are not incident on the same vertex then $G - e_1 - e_2$ is bipartite and 1-regular; $G$ has three connected components and $G - e_1 - e_2$ has four. As a free loop is a connected component, an Euler system for a graph must contain all the graph’s free loops. In figures representing graphs, free loops are drawn as circles with no incident vertices.

Let $U$ be an undirected 4-regular graph with $V(U) = \{v_1, ..., v_n\}$. It need not be the universe of any link diagram; we use the letter $U$ only for notational consistency.

**Definition 1** [38] If $C$ is an Euler system for $U$ then the interlacement matrix $I(U, C)$ is the $n \times n$ matrix over $GF(2)$ whose $ij$ entry is 1 if and only if $i \neq j$ and $v_i$ and $v_j$ are interlaced in $C$, i.e., when we follow $C$ starting at $v_i$ we encounter $v_j$, then $v_i$, then $v_j$ again before finally returning to $v_i$.

**Definition 2** If $U$ has $c(U)$ connected components then the interlacement graph $I(U, C)$ is obtained by adjoining $c(U) - 1$ free loops to the simple graph with vertex-set $V(U)$ and adjacency matrix $I(U, C)$.

Definition 2 differs from the corresponding definition of [38] in its use of free loops. Observe that if $U$ is disconnected then $I(U, C)$ consists of diagonal blocks corresponding to the interlacement matrices of the connected components of $U$. The same occurs if $U$ is a “connected sum,” i.e., if $U$ has two subgraphs $U_1$ and $U_2$ such that $V(U) = V(U_1) \cup V(U_2)$ and $|E(U) - E(U_1) - E(U_2)| = 2$. The free loops in Definition 2 allow $I(U, C)$ to distinguish these two situations: the interlacement graph of the disjoint union of $U_1$ and $U_2$ has one more free loop than the interlacement graph of a connected sum of $U_1$ and $U_2$.

**Definition 3** [25] The $\kappa$-transform $C* v$ of an Euler system $C$ at a vertex $v$ is obtained by reversing one of the two $v$-to-$v$ paths within the Euler circuit of $C$ in the connected component of $U$ containing $v$. 
The interlacement graph $I(U, C * v)$ is obtained from $I(U, C)$ by toggling (i.e., changing) every adjacency involving neighbors of $v$ in $I(U, C)$; that is, if $x, y \notin \{v\}$ are two distinct neighbors of $v$ in $I(U, C)$, then $x$ and $y$ are adjacent in $I(U, C * v)$ if and only if they are not adjacent in $I(U, C)$.

If we apply a single $\kappa$-transformation to a directed Euler system of a 2-in, 2-out digraph $\bar{U}$ then the result is no longer compatible with the edge-directions of $\bar{U}$. Suppose $C$ is a directed Euler system in $\bar{U}$, and suppose $v$ and $w$ are two vertices of $\bar{U}$ that are interlaced with respect to $C$. Then $C$ contains a circuit $vC_1wC_2vC_3wC_4$. The $\kappa$-transform $C * v$ contains an undirected circuit $vC_1wC_2vC_3wC_4$; the overbars indicate that the paths $wC_4v$ and $vC_3w$ have been reversed. Then $C * v * w$ contains $vC_1wC_4vC_2wC_3$ and $C * v * v * v$ contains $vC_1wC_4vC_3wC_2$. The lack of overbars indicates that this last is a directed circuit of $\bar{U}$. Arratia, Bollobás and Sorkin \cite{1, 2} call the operation $C \mapsto C * v * w * v$ a transposition on the pair $vw$. Comparing $vC_1wC_2vC_3wC_4$ to $vC_1wC_4vC_3wC_2$, we see that $I(U, C * v * w * v)$ is obtained from $I(U, C)$ by interchanging the neighbors of $v$ and $w$, and toggling every adjacency between vertices $x, y \notin \{v, w\}$ such that $x$ is adjacent to $v$, $y$ is adjacent to $w$, and either $x$ is not adjacent to $w$ or $y$ is not adjacent to $v$. That is, $I(U, C * v * w * v)$ is obtained from the pivot $I(U, C)vw$ by interchanging the neighbors of $v$ and $w$. (Simply interchanging the names of $v$ and $w$ would have the same effect, but we prefer not to do this as it would complicate our discussion later; see for instance Lemma \cite{3} below.)

A natural way to identify an Euler system $C$ in a 4-regular graph $U$ is to record, for each vertex of $U$, which of the three possible transitions (matchings of the incident half-edges) is used by $C$. This gives a rather complicated description, listing a pair of incident half-edges at each vertex. It is much simpler to use transitions to compare two Euler systems $C$ and $C'$. If we choose orientations for the Euler circuits in $C$ then these orientations allow us to make $U$ into a 2-in, 2-out digraph $\bar{U}$: at each vertex $v$, $C'$ must have either the same transition as $C$, the other transition that is compatible with the edge-directions in $\bar{U}$, or the transition that is incompatible with the edge-directions of $\bar{U}$. We say $C'$ follows $C$ at $v$, $C'$ is compatible with $C$ without following it at $v$, or $C'$ is incompatible with $C$ at $v$, respectively, to describe the three cases. The same description of the relationship between the transitions of $C$ and $C'$ at $v$ will hold if some circuits of $C$ are oriented differently. This simple description of the relationship between $C$ and $C'$ is useful in proving the following result, due to Kotzig \cite{25}, Ukkonen \cite{45} and Pevzner \cite{37}.

**Proposition 4** Every two Euler systems $C$ and $C'$ of a 4-regular undirected graph $U$ are connected to each other through a sequence of $\kappa$-transformations. If there is no vertex at which $C'$ is incompatible with $C$, they are connected through a sequence of transpositions.

**Proof.** If $C$ and $C'$ have the same transition at every vertex, then $C = C'$. 

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Proceeding by induction on the number of vertices at which their transitions differ, we have two cases.

If their transitions are incompatible at a vertex \( v \) then let \( C'' = C \ast v \). \( C'' \) has the same transitions as \( C \) except at \( v \), where it shares the transition of \( C' \), so the inductive hypothesis gives the result.

Suppose instead that there is no vertex at which \( C \) and \( C' \) have incompatible transitions. Let \( W \) denote the subset of \( V(U) \) containing the vertices at which the transitions of \( C \) and \( C' \) differ, and let \( w \in W \) have the shortest possible \( w \)-to-\( w \) path within \( C \); let the Euler circuit appearing in \( C \) that contains \( w \) be \( wC_1wC_2 \), with \( C_1 \) no longer than \( C_2 \). If no vertex of \( W \) appears on \( C_1 \) the transitions of \( C' \) are the same as those of \( C \) at every vertex of \( wC_1 \) except \( w \); then the circuit \( wC_1 \) appears in \( C' \), an impossibility as \( wC_1 \) is not an Euler circuit. The choice of \( w \) implies that no vertex of \( W \) can appear twice on \( C_1 \); hence there is a vertex \( w' \in W \) that appears precisely once on \( C_1 \). That is, \( w \) and \( w' \) are interlaced with respect to \( C \). Let \( C'' = C \ast w \ast w' \ast w \). Then \( C'' \) has the same transitions as \( C \) at all vertices except \( w \) and \( w' \), where it is compatible with \( C \); hence \( C'' \) has the same transitions at \( w \) and \( w' \) as \( C' \), has different transitions from \( C' \) at two fewer vertices than \( C \), and is not incompatible with \( C' \) at any vertex. The inductive hypothesis gives the result.

Just as using transitions to describe the relationship between two Euler systems is simpler than using transitions to describe a single Euler system, it is easier to describe an Euler system in the universe of a link diagram than it is to describe an Euler system in an arbitrary 4-regular graph. Suppose \( D \) is a diagram of a classical or virtual link \( L \), given with orientations of its components; these orientations make \( U \) into a 2-in, 2-out digraph \( \vec{U} \). (Any link component that is crossing-free in \( D \) appears as a free loop in \( U \).) At each crossing in \( D \), there are three transitions: one followed by the component(s) of \( L \), one that does not follow the components of \( L \) but is consistent with the edge-directions of \( \vec{U} \), and a third that is inconsistent with these edge-directions. A directed Euler system for \( \vec{U} \) will not involve any transitions of the third type. A simple description of a directed Euler system \( C \) is incorporated into a link diagram \( D \) by using the letter \( c \) to mark each crossing at which \( C \) is orientation-consistent with \( L \) without following it. (Crossings at which \( C \) follows \( L \) are left unmarked.) See Figures 2 and 3 in the Introduction. We do not use the traditional notation for smoothings (with 0 instead of \( c \)), in order to avoid confusion between notation that identifies an Euler system in \( U \) and notation that identifies a new diagram obtained by modifying \( D \). The same system is used to mark the vertices in the interlacement graph \( \mathcal{I}(U, C) \), resulting in the marked interlacement graph \( \mathcal{I}(D, C) \).

As noted in the Introduction, the famous Jones-Tutte connection of [42] is related to a choice of edge-directions in \( U \) which is alternating in the sense that as one traverses a component of the link, one encounters a reversal in edge-direction each time one passes through a vertex. If \( U \) admits an alternating orientation as a 2-in, 2-out digraph then there is an alternating diagram \( D^{alt} \) with universe
$U$: at each classical crossing the arc with both edges directed inward is the underpassing arc in $D^{alt}$. Lemma 7 of [18] then tells us that $L$ is checkerboard colorable; [18] also tells us that not all virtual links are checkerboard colorable, so restricting attention to Euler systems that give alternating orientations would involve a significant loss of generality. One way to avoid this loss of generality is to use nonorientable atoms; see [29] for instance. Our approach avoids loss of generality in a different way, by using edge-directions that are consistent with the orientations of the components of $L$ instead of using alternating edge-directions. By the way, a simple induction on the number of virtual crossings shows that it is always possible to choose an alternating orientation for a virtual knot diagram, but the orientation may not correspond to a 2-in, 2-out digraph. For instance every alternating orientation of the virtual knot diagram from [18] indicated in Figure 4 produces a vertex of indegree 4 and a vertex of outdegree 4. Diagrams of multi-component virtual links need not have any alternating orientations at all; for instance consider a diagram with only one virtual crossing, which involves two different link components.

**Definition 5** Suppose we are given a diagram $D$ of an oriented classical or virtual link $L$, along with a directed Euler system $C$ for $U$. Then the looped interlacement graph $L(D, C)$ is obtained from $I(D, C)$ by attaching a loop at each vertex corresponding to a negative crossing.

Recall that if $v$ and $w$ are vertices of a graph $G$ then the **pivot** $G^{vw}$ is the graph obtained from $G$ by toggling (reversing) the adjacency between every pair of vertices $x, y \notin \{v, w\}$ such that $x$ is a neighbor of $v$, $y$ is a neighbor of $w$, and either $x$ is not a neighbor of $w$ or $y$ is not a neighbor of $x$. (Pivots have no effect on loops.) We observed above that if a 4-regular graph $U$ has vertices $v, w$ that are interlaced with respect to an Euler system $C$, then $I(U, C \ast v \ast w \ast v)$
Figure 5: The crossing on the left is negative.

is the graph obtained from $I(U, C)^{vw}$ by interchanging the neighbors of $v$ and $w$. This observation applies to looped interlacement graphs as follows.

**Lemma 6** Suppose $D$ is an oriented link diagram with a directed Euler system $C$, and $v, w$ are adjacent in the looped interlacement graph $L(D, C)$. Then $L(D, C^{vw} v w v)$ is the marked graph obtained from $L(D, C)^{vw}$ by toggling the marks on $v$ and $w$ (i.e., removing a $c$ at $v$ or $w$ if one is there, and adjoining a $c$ at $v$ or $w$ if one is not there), and interchanging the neighbors of $v$ and $w$.

**Proof.** At any vertex other than $v$ or $w$, $C^{vw} v w v$ has the same transition as $C$. At $v$ and $w$, $C^{vw} v w v$ is compatible with $C$ without following it. ■

### 3 Examples

Figure 6 illustrates some of the definitions given in Section 2. Each row of the figure has an oriented knot diagram on the left-hand side, the corresponding directed universe in the middle, and the resulting looped interlacement graph on the right. The first two examples are different marked versions of a diagram of the figure-eight knot; the transitions of the Euler circuits are indicated by the patterns of dashes in the corresponding universes. The third example is the well-known virtual knot of [24], and the fourth is a virtual unknot diagram; their directed universes are isomorphic abstract graphs. All Euler circuits in the third and fourth directed universes give rise to the same looped interlacement graph, so we have not bothered to mark the diagrams or indicate Euler circuits.

Note that Definition 2 does not require any free loops because all four diagrams are connected. If we were to consider instead the split diagram consisting of the entire left-hand side of the figure, then the looped interlacement graph would include the right-hand side together with three additional free loops.

### 4 The extended Cohn-Lempel equality

Cohn and Lempel [11] described the number of orbits in a finite set under a certain kind of permutation using the nullity of an associated binary matrix.
Figure 6: In the top two rows, different marked versions of the same diagram give rise to different Euler circuits and looped interlacement graphs. In the bottom two rows, different diagrams give rise to the same universe and looped interlacement graph.
When applied to a connected 2-in, 2-out digraph \( \tilde{U} \) their equality uses the nullity of a submatrix of an interlace matrix to calculate the number of circuits that appear in a given partition of the edge-set of \( \tilde{U} \) into directed circuits. This equality was implicitly rediscovered as part of the combinatorial theory of the interlace polynomial (Theorem 24 of [1] or [2]), and its usefulness in knot theory was pointed out by Soboleva [34]. Other equalities similar to that of Cohn and Lempel have been used by knot theorists: Zulli [40] used a formula that always refers to \( n \times n \) matrices, and a more general version was stated by Mellor [33] and Lando [26].

An extended version of the Cohn-Lempel equality that includes all of these and applies to arbitrary circuit partitions in arbitrary 4-regular graphs is proven in [43]. Let \( U \) be an undirected 4-regular graph with \( \chi(U) = \{v_1, \ldots, v_n\} \), and let \( C \) be an Euler system for \( U \). Choose an orientation for each circuit in \( C \), and let \( \tilde{U} \) be the 2-in, 2-out digraph obtained from \( U \) by directing all edges according to these orientations. Suppose \( P \) is a partition of \( E(U) \) into undirected circuits; \( P \) should also contain every free loop \( U \) might have. Suppose that we follow a directed edge \( e \) of \( \tilde{U} \) toward a vertex \( v_i \). If the circuit of \( P \) that contains \( e \) leaves \( v_i \) along the edge \( C \) uses to leave \( v_i \) after arriving along \( e \), we say \( P \) follows \( C \) through \( v_i \). If the circuit of \( P \) that contains \( e \) leaves \( v_i \) along the other edge directed away from \( v_i \), we say \( P \) is orientation-consistent at \( v_i \) but does not follow \( C \). The last possibility is that the circuit of \( P \) that contains \( e \) leaves \( v_i \) along the other edge directed toward \( v_i \); in this case we say \( P \) is orientation-inconsistent at \( v_i \). (Note that the description of \( P \) at \( v_i \) is unchanged if we begin with the other edge directed toward \( v_i \), and it is not affected by the choice of orientations for the circuits of \( C \) or \( P \).) A matrix \( I_P = I_P(U, C) \) is associated to \( P \): if \( P \) follows \( C \) through \( v_i \) then the row and column of \( I(U, C) \) corresponding to \( v_i \) are removed; if \( P \) is orientation-consistent at \( v_i \) but does not follow \( C \) then the row and column of \( I(U, C) \) corresponding to \( v_i \) are retained; and if \( P \) is orientation-inconsistent at \( v_i \) then the off-diagonal entries of the row and column of \( I(U, C) \) corresponding to \( v_i \) are retained but their common diagonal entry is changed from 0 to 1.

**Proposition 7** (Extended Cohn-Lempel equality) Let \( U \) be an undirected, 4-regular graph with \( \chi(U) \) connected components, and let \( P \) be a partition of \( E(U) \) into undirected circuits; \( P \) must also contain every free loop of \( U \). Then the \( GF(2) \)-nullity of \( I_P \) is

\[
\nu(I_P) = |P| - \chi(U).
\]

We do not give a proof here; the interested reader can find one in [43]. Here is an example, though. Consider the complete bipartite graph \( K_{4,4} \), with vertices denoted 1, 2, 3, 4, 5, 6, 7 and 8; vertex \( i \) is adjacent to vertex \( j \) if and only if \( i \neq j \) (mod 2). Let \( C \) contain the Euler circuit 123456781475836. If \( P \) follows \( C \) at vertices 1 and 4, is orientation-inconsistent at vertices 2, 5 and 7, and is
orientation-consistent but does not follow $C$ at vertices 3, 6 and 8 then

$$\nu(I_P(U, C)) = \nu \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} = 0,$$

so $|P| = 1$. The one circuit in $P$ is the Euler circuit 1278345236741856. The partition $P'$ that disagrees with $P$ only by following $C$ at 2 and 6 corresponds to the matrix $I_{P'}(U, C)$ obtained by removing the first and fourth rows and columns of $I_P(U, C)$, so $\nu(I_{P'}(U, C)) = 2$. $P'$ contains the circuits 1236, 147658 and 254387.

## 5 A bracket polynomial for marked graphs

**Definition 8** If $G$ is a graph with $V(G) = \{v_1, ..., v_n\}$ then the Boolean adjacency matrix of $G$ is the $n \times n$ matrix $A(G)$ with the following entries in $GF(2)$: $A(G)_{ii} = 1$ if and only if $v_i$ is looped, and if $i \neq j$ then $A(G)_{ij} = 1$ if and only if $v_i$ is adjacent to $v_j$.

**Definition 9** A marked graph is a graph $G$ given with a partition of $V(G)$ into two subsets.

In our discussion we call the vertices in one cell of the partition unmarked, and vertices in the other cell marked (or marked with the letter c).

**Definition 10** Suppose $G$ is a marked graph with $V(G) = \{v_1, ..., v_n\}$ and $T \subseteq V(G)$. Let $\Delta_T$ be the $n \times n$ diagonal matrix whose $i^{th}$ diagonal entry is 1 if and only if $v_i \in T$. Then $A(G)_T$ is the submatrix of $A(G) + \Delta_T$ obtained by removing the $i^{th}$ row and column if $v_i$ is marked and the $i^{th}$ diagonal entry of $A(G) + \Delta_T$ is 0.

**Definition 11** The marked-graph bracket polynomial of a marked graph $G$ with $\phi$ free loops is

$$[G] = d^\phi \cdot \sum_{T \subseteq V(G)} A^{n-|T|} B^{|T|} d^{\nu(A(G)_T)},$$

where $\nu$ denotes the nullity of matrices with entries in $GF(2)$.

A simple consequence of Definitions [10] and [11] is that the marked-graph bracket is multiplicative on disjoint unions.

**Proposition 12** If $G$ is the union of disjoint subgraphs $G_1$ and $G_2$ then $[G] = [G_1] \cdot [G_2]$.
Proof. $G$ has $\phi = \phi_1 + \phi_2$ free loops. Also, if $T \subseteq V(G)$ and $T_i = T \cap V(G_i)$ then
\[
A(G)_T = \begin{pmatrix} A(G)_{T_1} & 0 \\ 0 & A(G)_{T_2} \end{pmatrix}
\]
so $\nu(A(G)_T) = \nu(A(G)_{T_1}) + \nu(A(G)_{T_2})$. ■

Another simple property of the marked-graph bracket is that toggling loops in $G$ has the effect of reversing the roles of $A$ and $B$ in $[G]$.

**Proposition 13** Let $G+I$ denote the marked graph obtained from $G$ by toggling all loops in $G$ (i.e., $G+I$ has loops at precisely those vertices where $G$ does not). Then $[G+I](A, B, d) = [G](B, A, d)$.

**Proof.** If $T \subseteq V(G)$ then $A(G)_T = A(G+I)_{V(G)-T}$, so the contribution of $T$ to $[G+I](A, B, d)$ is the same as the contribution of $V(G)-T$ to $[G](B, A, d)$. ■

Recall that the Kauffman bracket polynomial of a classical or virtual link diagram $D$ is a sum indexed by the states of $D$:
\[
[D] = \sum S A^{a(S)} B^{b(S)} d^{c(S)-1}.
\]
A Kauffman state $S$ corresponds in an obvious way to a partition $P(S)$ of $E(U)$ into undirected circuits, with the proviso that every free loop of $U$ is included in both $S$ and $P(S)$. If $C$ is a directed Euler system for $\vec{U}$ then we associate with each Kauffman state $S$ of $D$ the subset $T(S) \subseteq V(\mathcal{L}(D, C))$ consisting of the vertices corresponding to crossings where the state $S$ involves the $B$ smoothing. Observe that at an unmarked positive crossing, the $B$ smoothing corresponds to the transition that is orientation-inconsistent with $C$, and the $A$ smoothing corresponds to the transition that is orientation-consistent with $C$ without following $C$; at an unmarked negative crossing the $B$ smoothing is orientation-consistent with $C$ without following it, and the $A$ smoothing is orientation-inconsistent with $C$. At a positive crossing marked $c$ the $B$ smoothing is orientation-inconsistent with $C$ and the $A$ smoothing follows $C$; at a negative crossing marked $c$ the $B$ smoothing follows $C$ and the $A$ smoothing is orientation-inconsistent with $C$. Consequently if $S$ is a Kauffman state of $D$ then $A(\mathcal{L}(D,C))_{T(S)}$ is the same as the matrix denoted $I_{P(S)}(U,C)$ in the preceding section. The extended Cohn-Lempel equality then tells us that $|P(S)| = c(S) = \nu(A(\mathcal{L}(D,C))_{T(S)} + c(U)$. As $\mathcal{L}(D,C)$ has $\phi = c(U) - 1$ free loops, we conclude the following.

**Proposition 14** Suppose $D$ is a diagram of an oriented classical or virtual link $L$, $U$ is the universe of $D$ and $C$ is a directed Euler system for $\vec{U}$. Then the Kauffman bracket polynomial of $D$ and the marked-graph bracket polynomial of $\mathcal{L}(D,C)$ are the same.
Figure 7: A marked pivot toggles the marks on two adjacent vertices, and changes some adjacencies involving those vertices and their neighbors.

It follows that $[\mathcal{L}(D, C)]$ is independent of the choice of the particular directed Euler system $C$. This independence actually follows from Proposition $\Box$ and a more general result given in Theorem $\blacksquare$.

**Definition 15** Suppose $G$ is a marked graph and $v, w$ are adjacent in $G$. The **marked pivot** $G_{vw}^c$ is the marked graph obtained from $G_{vw}$ by toggling the marks on $v$ and $w$ separately (i.e., removing a mark $c$ where there is one, and inserting a mark $c$ where there is none), and interchanging the neighbors of $v$ and $w$.

Like an ordinary pivot, a marked pivot has no effect on loops or free loops.

**Theorem 16** Suppose $G$ is a marked graph and $v, w$ are adjacent in $G$. Then $[G] = [G_{vw}^c]$.

**Proof.** For convenience we presume $V(G) = \{v_1, ..., v_n\}$ with $v_1 = v$ and $v_2 = w$. The proposition is proven by showing that the nullities of $A(G)_T$ and $A(G_{vw}^c)_T$ are the same for every subset $T \subseteq V(G)$.

Suppose the first two diagonal entries of $A(G) + \Delta_T$ are both 1. Then

$$A(G)_T = A(G) + \Delta_T = \begin{pmatrix}
1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 \\
1 & M_{11} & M_{12} & M_{13} & M_{14} \\
1 & M_{21} & M_{22} & M_{23} & M_{24} \\
0 & M_{31} & M_{32} & M_{33} & M_{34} \\
0 & M_{41} & M_{42} & M_{43} & M_{44}
\end{pmatrix}$$

for appropriate submatrices $M_{ij}$, where a bold numeral indicates a row or column whose entries are all equal. Adding the first row of $A(G)_T$ to every row in the third and fifth blocks of rows (those containing $M_{11}$ and $M_{31}$), and then adding the second row to every row in the third and fourth blocks of rows, we
conclude that the nullity of $A(G_T)$ is the same as that of

$$
\begin{pmatrix}
1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & M_{11} & M_{12} & M_{13} & M_{14} \\
0 & 1 & M_{21} & M_{22} & M_{23} & M_{24} \\
1 & 0 & M_{31} & M_{32} & M_{33} & M_{34} \\
0 & 0 & M_{41} & M_{42} & M_{43} & M_{44}
\end{pmatrix},
$$

which is $A(G_{vw}^c)^T$ up to permutation of the rows and columns.

Suppose now that one of the first two diagonal entries of $A(G) + \Delta_T$ is 1, and the other is 0. If $v$ and $w$ are both unmarked then

$$
A(G)_T = A(G) + \Delta_T = 
\begin{pmatrix}
0 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & M_{11} & M_{12} & M_{13} & M_{14} \\
1 & 0 & M_{21} & M_{22} & M_{23} & M_{24} \\
0 & 1 & M_{31} & M_{32} & M_{33} & M_{34} \\
0 & 0 & M_{41} & M_{42} & M_{43} & M_{44}
\end{pmatrix}.
$$

Adding the first row of $A(G)_T$ to every row in the third and fifth blocks of rows, and then adding the second row to every row in the third and fourth blocks, we conclude that the nullity of $A(G)_T$ is the same as that of

$$
\begin{pmatrix}
0 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & M_{11} & M_{12} & M_{13} & M_{14} \\
1 & 0 & M_{21} & M_{22} & M_{23} & M_{24} \\
0 & 1 & M_{31} & M_{32} & M_{33} & M_{34} \\
0 & 0 & M_{41} & M_{42} & M_{43} & M_{44}
\end{pmatrix}.
$$

Removing the first column and second row of this matrix does not change its nullity, and yields $A(G_{vw}^c)^T$ up to permutation of rows and columns. The argument can be reversed if $v$ and $w$ are both marked in $G$, as they are then both unmarked in $G_{vw}^c$.

Also, if precisely one of $v, w$ is marked in $G$ then the marked one will correspond to the diagonal entry 1 in one of $G, G_{vw}^c$ and the diagonal entry 0 in the other of $G, G_{vw}^c$. The argument just given applies again, possibly with the roles of $G$ and $G_{vw}^c$ reversed.

Suppose now that the first two diagonal entries of $A(G) + \Delta_T$ are both 0, and suppose that precisely one of $v, w$ ($v$, say) is marked in $G$. Then

$$
A(G) + \Delta_T = 
\begin{pmatrix}
0 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & M_{11} & M_{12} & M_{13} & M_{14} \\
1 & 0 & M_{21} & M_{22} & M_{23} & M_{24} \\
0 & 1 & M_{31} & M_{32} & M_{33} & M_{34} \\
0 & 0 & M_{41} & M_{42} & M_{43} & M_{44}
\end{pmatrix}.
$$
and

\[ A(G)_T = \begin{pmatrix}
0 & 1 & 0 & 1 & 0 \\
1 & M_{11} & M_{12} & M_{13} & M_{14} \\
0 & M_{21} & M_{22} & M_{23} & M_{24} \\
1 & M_{31} & M_{32} & M_{33} & M_{34} \\
0 & M_{41} & M_{42} & M_{43} & M_{44}
\end{pmatrix} \]

for appropriate submatrices \( M_{ij} \). Adding the first row of \( A(G)_T \) to every row in the second and third blocks of rows, and then adding the first column to every column in the second and third blocks of columns, we conclude that the nullity of \( A(G)_T \) is the same as that of

\[ \begin{pmatrix}
0 & 1 & 0 & 1 & 0 \\
1 & M_{11} & M_{12} & M_{13} & M_{14} \\
0 & M_{21} & M_{22} & M_{23} & M_{24} \\
1 & M_{31} & M_{32} & M_{33} & M_{34} \\
0 & M_{41} & M_{42} & M_{43} & M_{44}
\end{pmatrix}, \]

which is \( A(G_{vw}^c)_T \) up to a permutation of the rows and columns.

Suppose now that the first two diagonal entries of \( A(G) + \Delta_T \) are both 0 and that \( v \) and \( w \) are both unmarked in \( G \). Then

\[ A(G) + \Delta_T = A(G)_T = \begin{pmatrix}
0 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 \\
1 & 1 & M_{11} & M_{12} & M_{13} & M_{14} \\
1 & 0 & M_{21} & M_{22} & M_{23} & M_{24} \\
0 & 1 & M_{31} & M_{32} & M_{33} & M_{34} \\
0 & 0 & M_{41} & M_{42} & M_{43} & M_{44}
\end{pmatrix}. \]

Adding the first row of \( A(G)_T \) to every row in the third and fifth blocks of rows, and then adding the second row to every row in the third and fourth blocks, we conclude that the nullity of \( A(G)_T \) is the same as that of

\[ \begin{pmatrix}
0 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 \\
1 & 1 & M_{11} & M_{12} & M_{13} & M_{14} \\
1 & 0 & M_{21} & M_{22} & M_{23} & M_{24} \\
0 & 1 & M_{31} & M_{32} & M_{33} & M_{34} \\
0 & 0 & M_{41} & M_{42} & M_{43} & M_{44}
\end{pmatrix}. \]

Removing the first two rows and columns of this matrix does not change the nullity, and yields \( A(G_{vw}^c)_T \), up to a permutation of the rows and columns.

If the first two diagonal entries of \( A(G) + \Delta_T \) are both 0 and \( v \) and \( w \) are both marked the same argument applies, with the roles of \( G \) and \( G_{vw}^c \) reversed.

\[ \blacksquare \]

Theorem [16] will be helpful in the balance of the paper because it allows us to focus on a limited number of special cases. For instance, we will see
later that when applied to different configurations of marked vertices, the knot-theoretic Reidemeister moves give rise to different marked-graph Reidemeister moves. Theorem 16 allows us to ignore some of those configurations (e.g., ones involving adjacent marked vertices).

6 A recursion for the marked-graph bracket

In this section we present a recursive description of the marked-graph bracket polynomial. The recursion involves formulas that are only valid when certain vertices are without marked neighbors, so an arbitrary marked graph must be “prepared” for the recursion by using marked pivots to eliminate adjacencies between marked vertices.

Figure 8: Marked pivots can be used to remove marks on adjacent marked vertices. Different marked graphs may result, but they will all have the same bracket polynomial.

Definition 17 If $v$ is a vertex of a (marked) graph $G$ then the local complement $G^v$ is obtained from $G$ by toggling every edge incident only on neighbor(s) of $v$. 

21
Recall that \( v \) itself is not included among its neighbors, so edges incident on \( v \) in \( G \) are preserved in \( G^v \). We should also point out that there is a different use of the term “local complement,” which is restricted to simple graphs and does not involve loops [7, 8]; we follow [1, 2, 3] instead, and toggle both loops and non-loop edges incident on neighbors of \( v \) when constructing \( G^v \).

**Theorem 18** The marked-graph bracket polynomial of a marked graph \( G \) can be calculated recursively using the following steps.

(a) If \( v \) and \( w \) are marked neighbors, replace \( G \) with \( G_{vw} \).

(b) Suppose \( v \) is unlooped and marked, and no neighbor of \( v \) is marked. Then

\[
\]

where \( G - v \) is obtained from \( G \) by removing \( v \) and every edge incident on \( v \).

(c) Suppose \( v \) is looped and marked, and no neighbor of \( v \) is marked. Then

\[
\]

(d) If \( v \) is looped and no neighbor of \( v \) is marked, then

\[
\]

Here \( G - \{v, v\} \) is obtained from \( G \) by removing the loop at \( v \).

(e) Let \( v \) and \( w \) be adjacent, unlooped, unmarked vertices. If no neighbor of \( v \) is marked then

\[
\]

(f) Suppose \( G \) is a graph whose edges are all loops. If \( G \) has \( \phi \) free loops, \( n_1 \) unlooped marked vertices, \( n_2 \) looped marked vertices, \( n_1 \) unlooped unmarked vertices and \( n_2 \) looped unmarked vertices then

\[
[G] = d^\phi (A + B)^{n_1 + n_2} (Ad + B)^{n_1} (A + Bd)^{n_2}.
\]
Proof. Theorem 10 tells us that an application of part (a) does not change the marked-graph bracket polynomial, and the formula of part (f) follows directly from Definition 11. As free loops are not affected by local complementation, pivots or marked pivots, the corresponding powers of $d$ can simply be factored out of the formulas of parts (b)-(e); hence it suffices to verify (b)-(e) for graphs without free loops.

In part (b) $v$ is a marked, unlooped vertex with no marked neighbor. Let

$$S = \sum_{v \in T \subseteq V(G)} A^{n-|T|}B^{|T|}d^p(A(G)_T).$$

If $v \in T \subseteq V(G)$ then using elementary row operations, we see that

$$\nu(A(G)_T) = \nu\left(\begin{array}{ccc}
1 & 1 & 0 \\
1 & M_{11} & M_{12} \\
0 & M_{21} & M_{22}
\end{array}\right)$$

$$= \nu\left(\begin{array}{ccc}
1 & 1 & 0 \\
0 & M_{11} & M_{12} \\
0 & M_{21} & M_{22}
\end{array}\right) = \nu\left(\begin{array}{ll}
\bar{M}_{11} & \bar{M}_{12} \\
\bar{M}_{21} & \bar{M}_{22}
\end{array}\right)$$

for appropriate matrices $M_{ij}$. As $v$ has no marked neighbor, the toggling of diagonal entries in $\bar{M}_{11}$ is compatible with Definition 10 and we conclude that

$$\nu(A(G)_T) = \nu(A(G_v - v)_{T-{v}}).$$

This holds whenever $v \in T$, so $S = B[G_v - v]$. As $v$ is marked, Definition 10 has $A(G)_T = A(G - v)_T$ for every $T \subseteq V(G)$ with $v \notin T$. It follows that $[G] - S = A[G - v]$. Part (c) follows from part (b) and Proposition 13.

We proceed to consider (d). Suppose $G$ has a looped vertex $v$ whose neighbors are all unmarked. If $v \in T \subseteq V(G)$ then $A(G)_T = A(G - \{v, v\})_{T-{v}}$ and $A(G)_{T-{v}} = A(G - \{v,v\})_T$; these equalities hold whether or not $v$ is marked. Let

$$S = \sum_{v \in T \subseteq V(G)} A^{n-|T|}B^{|T|}d^p(A(G - \{v,v\})_T).$$


Turning to (e), suppose that $v, w \in V(G)$ are adjacent, unlooped, unmarked vertices, and no neighbor of $v$ is marked. Let

$$S_1 = \sum_{T \subseteq V(G) \atop v \notin T, w \notin T} A^{n-|T|}B^{|T|}d^p(A(G)_T)$$

and

$$S_2 = \sum_{T \subseteq V(G) \atop v \notin T, w \in T} A^{n-|T|}B^{|T|}d^p(A(G)_T).$$

23
As \( v \) has no marked neighbor the nullity argument given for part (b) applies; it
tells us that \( |G| - S_1 - S_2 = B[G^v - v] \).

Consider a subset \( T \subseteq V(G) \) with \( v, w \notin T \). Adding the first row to those
in the third and fifth blocks of rows and adding the second row to those in the
third and fourth blocks of rows, we see that

\[
\nu(A(G)_T) = \nu \begin{pmatrix}
0 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 \\
1 & 1 & M_{11} & M_{12} & M_{13} & M_{14} \\
1 & 0 & M_{21} & M_{22} & M_{23} & M_{24} \\
0 & 1 & M_{31} & M_{32} & M_{33} & M_{34} \\
0 & 0 & M_{41} & M_{42} & M_{43} & M_{44}
\end{pmatrix}
\]

\[
= \nu \begin{pmatrix}
0 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 \\
0 & 0 & M_{11} & M_{12} & M_{13} & M_{14} \\
0 & 0 & M_{21} & M_{22} & M_{23} & M_{24} \\
0 & 0 & M_{31} & M_{32} & M_{33} & M_{34} \\
0 & 0 & M_{41} & M_{42} & M_{43} & M_{44}
\end{pmatrix}
\]

\[
= \nu \begin{pmatrix}
M_{11} & M_{12} & M_{13} & M_{14} \\
M_{21} & M_{22} & M_{23} & M_{24} \\
M_{31} & M_{32} & M_{33} & M_{34} \\
M_{41} & M_{42} & M_{43} & M_{44}
\end{pmatrix}
\]

\[
= \nu(A(G^{vw} - v - w)_T).
\]

It follows that \( S_1 = A^2[G^{vw} - v - w] \).

Now, consider a subset \( T \subseteq V(G) \) with \( v \notin T \) and \( w \in T \). Adding the first
row to those in the fourth and fifth blocks of rows, and the second row to those
in the third and fourth blocks, we see that

\[
\nu(A(G)_{T}) = \nu
\begin{pmatrix}
0 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & M_{11} & M_{12} & M_{13} & M_{14} \\
1 & 0 & M_{21} & M_{22} & M_{23} & M_{24} \\
0 & 1 & M_{31} & M_{32} & M_{33} & M_{34} \\
0 & 0 & M_{41} & M_{42} & M_{43} & M_{44}
\end{pmatrix}
\]

\[
= \nu
\begin{pmatrix}
0 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & M_{11} & M_{12} & M_{13} & M_{14} \\
0 & 0 & M_{21} & M_{22} & M_{23} & M_{24} \\
0 & 0 & M_{31} & M_{32} & M_{33} & M_{34} \\
0 & 0 & M_{41} & M_{42} & M_{43} & M_{44}
\end{pmatrix}
\]

\[
= \nu
\begin{pmatrix}
\bar{M}_{11} & M_{12} & \bar{M}_{13} & M_{14} \\
M_{21} & M_{22} & M_{23} & M_{24} \\
\bar{M}_{31} & \bar{M}_{32} & M_{33} & M_{34} \\
M_{41} & M_{42} & M_{43} & M_{44}
\end{pmatrix}
\]

\[
= \nu(A((G'^{vw})^v - v - w)_{T-\{w\}}).
\]

The last equality is justified because \(v\) has no marked neighbors in \(G\), so the toggling of diagonal entries in \(M_{11}\) and \(M_{22}\) is consistent with Definition 10. It follows that \(S_2 = AB[(G'^{vw})^v - v - w]\). 

One way to implement Theorem 18 is to follow this simple outline: first use (a) to remove marks on adjacent vertices; then use (b) and (c) to remove marked vertices; then use (d) to remove loops; and finally use (e) to remove the remaining edges. Typically, this will not result in the most efficient recursive calculation of the bracket polynomial of a given marked graph. For one thing, an implementation of any branching algorithm can be simplified if one combines terms that correspond to isomorphic structures as they arise during the recursion. Also, in our case it is obviously convenient to use part (f) when \(G\) has only isolated vertices, rather than first using parts (c) and (d) to deal with those vertices that happen to be looped. More generally, Proposition 13 will sometimes yield a more convenient graph; for instance, if the vertices of \(G\) are all looped it is certainly simpler to unloop all the vertices at once rather than using parts (c) and (d) of Theorem 18 repeatedly. In addition, an efficient computation would utilize Proposition 12. The computational cost of an instance of part (b), (c), (d) or (e) varies according to the structure of the resulting graphs \(G^v, G'^{vw}\) and \((G'^{vw})^v\); there are generally many choices of vertices at which parts (b)-(e) might be applied, and it is not easy to guess which of these choices might be most convenient. Finally, there are other identities which might be used in particular situations (when two unmarked looped vertices are adjacent, when an unmarked looped vertex is adjacent to a marked unlooped vertex, etc.) and
might sometimes appear in more efficient computations than an implementation of Theorem 18. Two of these other identities are given in Corollary 19 and Proposition 20.

7 Applying Theorem 18 to links

Figure 10: A bracket calculation.

Figure 10 illustrates a calculation of the Kauffman bracket of a diagram of Whitehead’s link using Theorem 18. The marked link diagram $D$ appears at the top left of the figure, with $\mathcal{L}(D, C)$ at the top right. The two arrows directed downward from $\mathcal{L}(D, C)$ point to the two graphs that result when part (d) of Theorem 18 is applied, with $v$ the only vertex of $\mathcal{L}(D, C)$ that is not adjacent...
to the marked vertex. Part (e) of Theorem 18 is then applied to the left-hand graph in the middle row of the figure, with \( v \) again the only vertex of \( \mathcal{L}(D, C) \) that is not adjacent to the marked vertex. As indicated in the figure, the second term of the first step and the third term of the second step refer to the same graph \( G'' - v \). The bracket polynomial of this graph is calculated using part (c): 

\[
\]

where \( y \) is the marked vertex.

The result of the calculation is

\[
+ A(Ad + B)^3 + B(A^3d^2 + 3A^2Bd + 3AB^2 + B^2d)
\]

Another calculation of the same bracket polynomial is indicated in Figure 11. This time the first step is an application of part (c), \( [G] = B[G - v] + A[G'' - v] \) with \( v \) the marked vertex. The second step is an application of part (d), and the third step is an application of part (e). The result of the calculation is

\[
= BA^{-1}B \cdot (A^2(Ad + B)(A + Bd) + AB(Ad + B)^2)
+ BA^{-1}B \cdot B(A^3 + 3A^2Bd + AB^2d^2 + 2AB^2 + B^3d)
\]

The recursive calculation of the Kauffman bracket \([D] = \mathcal{L}(D, C)\) provided by Theorem 18 has not appeared before, but the individual steps are all related to familiar properties of the Kauffman bracket or Jones polynomial. The basic recursion of Kauffman’s bracket, \([D] = A[D_A] + B[D_B]\), is extended to marked graphs in parts (b) and (c) of Theorem 18. As discussed in Section 4 of [4], the formula of part (d) of Theorem 18 corresponds to the Jones polynomial’s braid-plat formula (denoted \( tV_{n-1} - V_1 = t^{\beta_{1}}(t-1)\bar{V}_1 \) in [4]) and the Kauffman bracket’s switching formula (denoted \( A_{\chi} - A_{-\chi} = (A^2 - A^{-2}) \propto \bar{\chi} \) in [22]), and the formula of part (e) corresponds to a double use of the basic recursion of the Kauffman bracket, \([D] = A^2[D_{AA}] + AB[D_{AB}] + B[D_B]\).

In addition, Theorem 18 implies the reversing property of the Jones polynomial [27, 31]. The proof is simple: consider a link diagram in which a certain component is incident on only one marked crossing, and apply Corollary 19 to conclude that the Kauffman bracket is not changed when the orientation of that component is reversed.

**Corollary 19** Suppose \( v \) is looped and marked in \( G \), and no neighbor of \( v \) is marked. Then \([G] = [G' - \{v, v\}]\).

**Proof.** This follows immediately from parts (b) and (c) of Theorem 18. \( \blacksquare \)
Figure 11: Another bracket calculation.
The recursion of Theorem 18 is fundamentally different from the original recursive description of the classical Jones polynomial [17], which involves reducing the unknotting number (a quantity that is not generally well defined for virtual links). The basic formula $t^{-1}V_{L^+} - tV_{L^-} = (t^{1/2} - t^{-1/2})V_L$ of [17] corresponds to one of the “other identities” mentioned in the preceding section:

**Proposition 20** If $v \in V(G)$ is looped and marked then

$$[G] = AB^{-1}[G - \{v, v\}] + (B - A^2B^{-1})[G - v].$$

**Proof.** If $v \in T \subseteq V(G)$ then $A(G)_T = A(G - \{v, v\})_{T - \{v\}}$ and $A(G)_{T - \{v\}} = A(G - \{v, v\})_T$. Consequently, if

$$S = \sum_{v \in T \subseteq V(G)} A^{n - |T|}B^{|T|}d^v(A(G)_T)$$


Observe that unlike the formulas given in Theorem 18, Proposition 20 allows $v$ to have marked neighbors.

### 8 $\Omega.2$ moves and the reduced bracket

**Definition 21** If $G$ is a marked graph then the reduced marked-graph bracket $\langle G \rangle$ is obtained from $[G]$ by the evaluations $B \mapsto A^{-1}$ and $d \mapsto -A^2 - A^{-2}$.

**Proposition 22** Suppose $G$ is a marked graph with two unmarked vertices $v$ and $w$ such that $v$ is looped, $w$ is not, and $v$ and $w$ are adjacent to the same vertices outside $\{v, w\}$. Then $\langle G \rangle = \langle G - v - w \rangle$.

**Proof.** Suppose for convenience that $V(G) = \{v_1, ..., v_n\}$ with $v_1 = v$ and $v_2 = w$. If $T \subseteq V(G) - \{v, w\}$ then let $T_1 = T \cup \{v\}$, $T_2 = T \cup \{w\}$ and $T_{12} = T \cup \{v, w\}$.

If $v$ and $w$ are not adjacent then using elementary row and column operations...
we see that

$$\nu(A(G)_T) = \nu \left( \begin{array}{ccc} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & M_{11} \\ 0 & 0 & M_{21} \end{array} \right) = \nu \left( \begin{array}{ccc} 0 & 1 & 0 \\ 1 & M_{11} & M_{12} \\ 0 & M_{21} & M_{22} \end{array} \right),$$

$$\nu(A(G)_{T_1}) = \nu \left( \begin{array}{ccc} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & M_{11} \\ 0 & 0 & M_{21} \end{array} \right) = 1 + \nu \left( \begin{array}{ccc} 0 & 1 & 0 \\ 1 & M_{11} & M_{12} \\ 0 & M_{21} & M_{22} \end{array} \right),$$

$$\nu(A(G)_{T_2}) = \nu \left( \begin{array}{ccc} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & M_{11} \\ 0 & 0 & M_{21} \end{array} \right) = \nu \left( \begin{array}{ccc} M_{11} & M_{12} \\ M_{21} & M_{22} \end{array} \right) \text{ and}$$

$$\nu(A(G)_{T_{12}}) = \nu \left( \begin{array}{ccc} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & M_{11} \\ 0 & 0 & M_{21} \end{array} \right) = \nu \left( \begin{array}{ccc} 0 & 1 & 0 \\ 1 & M_{11} & M_{12} \\ 0 & M_{21} & M_{22} \end{array} \right).$$

It follows that the image of $A^2d^{\nu(A(G)_T)} + ABd^{\nu(A(G)_{T_1})} + B^2d^{\nu(A(G)_{T_{12}})}$ under the evaluations $B \mapsto A^{-1}$ and $d \mapsto -A^2 - A^{-2}$ is 0. The image of the remaining term is the contribution of $T$ to the reduced bracket of $G - v - w$, so $\langle G \rangle = \langle G - v - w \rangle$.

If $v$ and $w$ are adjacent then

$$\nu(A(G)_T) = \nu \left( \begin{array}{ccc} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & M_{11} \\ 0 & 0 & M_{21} \end{array} \right) = \nu \left( \begin{array}{ccc} 1 & 1 & 0 \\ 1 & M_{11} & M_{12} \\ 0 & M_{21} & M_{22} \end{array} \right),$$

$$\nu(A(G)_{T_1}) = \nu \left( \begin{array}{ccc} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & M_{11} \\ 0 & 0 & M_{21} \end{array} \right) = \nu \left( \begin{array}{ccc} M_{11} & M_{12} \\ M_{21} & M_{22} \end{array} \right),$$

$$\nu(A(G)_{T_2}) = \nu \left( \begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & M_{11} \\ 0 & 0 & M_{21} \end{array} \right) = 1 + \nu \left( \begin{array}{ccc} 1 & 1 & 0 \\ 1 & M_{11} & M_{12} \\ 0 & M_{21} & M_{22} \end{array} \right), \text{ and}$$

$$\nu(A(G)_{T_{12}}) = \nu \left( \begin{array}{ccc} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & M_{11} \\ 0 & 0 & M_{21} \end{array} \right) = \nu \left( \begin{array}{ccc} 1 & 1 & 0 \\ 1 & M_{11} & M_{12} \\ 0 & M_{21} & M_{22} \end{array} \right).$$

The image of $A^2d^{\nu(A(G)_T)} + ABd^{\nu(A(G)_{T_1})} + B^2d^{\nu(A(G)_{T_{12}})}$ under the evaluations $B \mapsto A^{-1}$ and $d \mapsto -A^2 - A^{-2}$ is 0, and the image of the remaining term is the contribution of $T$ to the reduced bracket of $G - v - w$. ■
Proposition 23 Suppose $G$ is a marked graph with two adjacent vertices $v$ and $w$ such that $v$ is looped and marked, $w$ is neither looped nor marked, and $v$ is the only neighbor of $w$. If $v$ has no neighbor other than $w$ then $\langle G \rangle = \langle G^+ - v - w \rangle$, where $G^+$ is obtained from $G$ by adjoining a free loop. Otherwise, let $z$ be a neighbor of $v$ other than $w$. Then $G = \langle G^2_{cz} - v - w \rangle$.

Proof. If $v$ has no neighbor other than $w$ then Proposition 16 tells us that $\langle G \rangle = \langle G_{cz}^2 \rangle$. In $G_{cz}^2$, $v$ and $w$ are adjacent, unmarked vertices, with $v$ looped and $w$ unlooped. When constructing $G_{cz}^2$ we first construct $G_{cz}^w$, by toggling all adjacencies between vertices $x, y \notin \{v, z\}$ such that $x$ is adjacent to $v$, $y$ is adjacent to $z$ and either $x$ is not adjacent to $z$ or $y$ is not adjacent to $v$. This includes toggling every adjacency between $w$ and a neighbor of $z$ other than $v$. As $w$ is adjacent to none of these vertices in $G$, and not adjacent to any other vertex aside from $v$, the neighbors of $w$ in $G_{cz}^w$ are the same as the neighbors of $z$. The construction of $G_{cz}^w$ is completed by interchanging the neighbors of $w$ and $z$, so the neighbors of $w$ in $G_{cz}^w$ are the same as the neighbors of $v$ in $G_{cz}^w$. Proposition 22 then tells us that $\langle G_{cz}^w \rangle = \langle G_{cz}^w - v - w \rangle$.

Proposition 24 Suppose $G$ is a marked graph with two adjacent vertices $v$ and $w$ such that $v$ is looped and marked, $w$ is unlooped and unmarked, and $v$ and $w$ have precisely the same neighbors outside $\{v, w\}$. If $v$ has no neighbor other than $w$ then $\langle G \rangle = \langle G^+ - v - w \rangle$. Otherwise, $\langle G \rangle = \langle G^2_{cw} - v - w \rangle$ for any $z \neq w$ that is adjacent to $v$.

Proof. If $v$ has no neighbor other than $w$ the preceding proposition applies.

Suppose $z$ is a neighbor of $v$ other than $w$; then $\langle G \rangle = \langle G_{cw}^z \rangle$. In $G_{cw}^z$, the adjacency between $w$ and every vertex $x \notin \{v, z\}$ that is adjacent to only one of $v, z$ is toggled; as $w$ has the same neighbors outside $\{v, w\}$ as $v$ has, it follows that $w$ has the same neighbors in $G_{cv}^z$ as $z$ has. The neighbors of $v$ and $z$ in $G_{cv}^z$ are exchanged in $G_{cw}^z$, so Proposition 22 applies to $v$ and $w$ in $G_{cw}^z$.

Definition 25 A marked-graph $\Omega.2$ Reidemeister move is constructed as follows. Any finite sequence of marked pivots may be applied to a marked graph $H$; let $G$ denote the result. $G$ is replaced with $G - v - w$, $G^+ - v - w$ or $G_{cz}^2 - v - w$ as in one of the three propositions above. Any finite sequence of marked pivots may then be applied. The inverse of an $\Omega.2$ move is also an $\Omega.2$ move, as is the move obtained by first toggling all loops in $H$, and then toggling all loops in the final graph.

Proposition 16 and Theorem 16 tell us that if $G'$ is obtained from $G$ by using a marked-graph $\Omega.2$ move to adjoin or remove two vertices, then $\langle G \rangle = \langle G' \rangle$. 

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Recall that if $D$ is a link diagram then an $\Omega.2$ Reidemeister move on $D$ involves replacing one of the two configurations pictured on the left or right of Figure 12 with the configuration in the middle.

**Proposition 26** If $D'$ is obtained by applying an $\Omega.2$ Reidemeister move to a link diagram $D$ then there are Euler systems $C$ and $C'$ such that $\mathcal{L}(D', C')$ is obtained from $\mathcal{L}(D, C)$ by applying a marked-graph $\Omega.2$ move.

**Proof.** Suppose first that $D'$ is obtained from $D$ by replacing the configuration on the left in Figure 12 with the configuration that appears in the middle, and $C$ is an Euler system for the directed universe $\hat{U}$ of $D$. There are four possible arrangements of marks: either of $v, w$ could be marked, or neither, or both. In every one of these configurations the vertices of $U$ corresponding to $v$ and $w$ are interlaced with respect to $C$, so if $w$ is marked we may replace $C$ with $C * v * w * v$. We proceed under the assumption that $w$ is not marked.

If neither $v$ nor $w$ is marked, the Euler system $C$ must contain a circuit that leaves $w$ along the edge corresponding to the pictured arc $e$, and whose next appearance in the pictured portion of $U$ occurs along the edge corresponding to the pictured arc $g$. Then the crossings $v$ and $w$ correspond to adjacent, unmarked vertices $v, w \in V(U)$ such that such that $v$ is looped, $w$ is unlooped, and $v$ and $w$ are adjacent to the same vertices outside $\{v, w\}$. Removing $v$ and $w$ from $\mathcal{L}(D, C)$ is an instance of Proposition 22 and results in $\mathcal{L}(D', C')$, where $C'$ is the Euler system for $D'$ obtained in the obvious way from $C$.

If $v$ is marked and $w$ is not, $C$ must instead contain a circuit $\gamma$ that leaves the portion of $D$ pictured in Figure 12 along $e$ and re-enters along $h$. The crossings $v$ and $w$ again correspond to adjacent vertices $v, w \in V(U)$ such that $v$ is looped, $w$ is unlooped, and $v$ and $w$ are adjacent to the same vertices outside $\{v, w\}$. If $\gamma$ passes through no crossing of $D$ only once on the way from $e$ to $h$, then $v$ and $w$ have no neighbor in $\mathcal{L}(D, C)$ except each other. Removing $v$ and $w$ is then an
instance of the first clause of Proposition 23, and \( L(D, C) - v - w = L(D', C') \) where \( C' \) is the Euler system for \( D' \) that contains the \( e \)-to-\( h \) and \( f \)-to-\( g \) portions of \( \gamma \) as separate circuits. If \( \gamma \) passes through a crossing \( z \) only once on the way from \( e \) to \( h \), then Lemma 6 tells us that \( L(D, C)'(v, z) = L(D, C* v * z * v) \). As \( v \) and \( w \) are both unmarked in \( L(D, C)'(v, z) \), we may refer to the preceding paragraph.

Suppose now that \( D' \) is obtained from \( D \) by replacing the configuration on the right in Figure 12 with the configuration in the middle. It is impossible for both \( v \) and \( w \) to be marked, for this would result in an “extra” circuit incident on only \( v \) and \( w \); as \( C \) is an Euler system, it cannot contain such a circuit.

If neither \( v \) nor \( w \) is marked, \( C \) contains a circuit that leaves \( w \) along the edge corresponding to \( f \) and then re-enters the portion of \( D \) pictured in Figure 12 at \( e \). Then \( v \) is looped, \( w \) is unlooped, and \( v, w \) are nonadjacent vertices with the same neighbors. Removing \( v \) and \( w \) from \( L(D, C) \) is an instance of Proposition 22, and the result is \( L(D', C') \) where \( C' \) is the Euler system obtained from \( C \) in the obvious way.

If \( v \) is marked and \( w \) is not then \( v \) is the only neighbor of \( w \) in \( L(D, C) \). As in the third paragraph, we have two cases. If \( w \) is the only neighbor of \( v \) in \( L(D, C) \) then we refer to the first clause of Proposition 23 \( L(D, C) - v - w = L(D', C') \) where \( C' \) is obtained from \( C \) by replacing the circuit \( \gamma \) that passes through \( v \) and \( w \) with two separate circuits, one including the \( h \)-to-\( e \) portion of \( \gamma \) and the other including the \( f \)-to-\( g \) portion of \( \gamma \). If instead \( v \) has a neighbor \( z \neq w \) in \( L(D, C) \), then we replace \( C \) by \( C* v * z * v \) and refer to the preceding paragraph.

If \( w \) is marked and \( v \) is not then we toggle all loops, apply the preceding paragraph with the roles of \( v \) and \( w \) reversed, and then toggle all loops again.

\section{\( \Omega .3 \) moves and the reduced bracket}

**Proposition 27** Suppose \( G \) is a marked graph with three unmarked vertices \( u, v, w \) such that \( u, v, w \) are all adjacent to each other, \( u \) is looped, \( v \) and \( w \) are unlooped, and every vertex \( x \notin \{u, v, w\} \) is adjacent to either 0 or precisely two of \( u, v, w \). Let \( G' \) be the graph obtained from \( G \) by removing all three edges \( \{u, v\}, \{u, w\} \) and \( \{v, w\} \). Then \( \langle G \rangle = \langle G' \rangle \).

**Proof.** Each \( T \subseteq V(G) - \{u, v, w\} \) corresponds to eight subsets of \( V(G) \), which make eight contributions to \( \langle G \rangle \) and eight contributions to \( \langle G' \rangle \). In each case, three of the eight cancel. The proposition is true because the remaining five happen to equal each other.

Considering \( G \) first, observe that the \( GF(2) \)-nullities of \( A(G)_T, A(G)_{T \cup \{u\}} \)
and $\mathcal{A}(G)_{T \cup \{u,v\}}$ are (respectively)

$$
\nu
\begin{pmatrix}
\kappa_1 & \kappa_2 & \kappa_1 + \kappa_2 & M \\
1 & 1 & 1 & \rho_1 \\
1 & 0 & 1 & \rho_2 \\
1 & 1 & 0 & \rho_1 + \rho_2 \\
\end{pmatrix}
= \nu
\begin{pmatrix}
\kappa_1 & \kappa_2 & \kappa_1 + \kappa_2 & M \\
1 & 1 & 1 & \rho_1 \\
1 & 0 & 1 & \rho_2 \\
1 & 1 & 0 & \rho_1 + \rho_2 \\
\end{pmatrix}
= \nu
\begin{pmatrix}
\kappa_1 & \kappa_2 & \kappa_1 + \kappa_2 & M \\
0 & 1 & 0 & \rho_1 + \rho_2 \\
0 & 0 & 1 & \rho_1 \\
0 & 0 & 1 & \rho_1 \\
\end{pmatrix}
$$

and

$$
\nu
\begin{pmatrix}
\kappa_1 & \kappa_2 & \kappa_1 + \kappa_2 & M \\
0 & 1 & 1 & \rho_1 \\
1 & 1 & 1 & \rho_2 \\
1 & 1 & 0 & \rho_1 + \rho_2 \\
\end{pmatrix}
= \nu
\begin{pmatrix}
\kappa_1 & \kappa_2 & \kappa_1 + \kappa_2 & M \\
1 & 0 & 0 & \rho_1 \\
0 & 1 & 0 & \rho_1 + \rho_2 \\
0 & 0 & 1 & \rho_1 \\
\end{pmatrix}
$$

for appropriate row vectors $\rho_i$, column vectors $\kappa_i$ and a submatrix $M$. Consequently $\nu(\mathcal{A}(G)_T) = \nu(\mathcal{A}(G)_{T \cup \{u,v\}}) = \nu(\mathcal{A}(G)_{T \cup \{u\}}) - 1$; it follows that the contributions of $T$, $T \cup \{u\}$ and $T \cup \{u,v\}$ to $\langle G \rangle$ cancel each other.

Similarly, the contributions of $T$, $T \cup \{u\}$ and $T \cup \{u,v\}$ to $\langle G' \rangle$ all cancel each other.

The $GF(2)$-nullities of $\mathcal{A}(G)_{T \cup \{v\}}$ and $\mathcal{A}(G')_{T \cup \{v\}}$ are

$$
\nu
\begin{pmatrix}
\kappa_1 & \kappa_2 & \kappa_1 + \kappa_2 & M \\
1 & 1 & 1 & \rho_1 \\
1 & 1 & 1 & \rho_2 \\
1 & 1 & 0 & \rho_1 + \rho_2 \\
\end{pmatrix}
= \nu
\begin{pmatrix}
\kappa_1 & \kappa_2 & \kappa_1 + \kappa_2 & M \\
0 & 1 & 0 & \rho_1 \\
0 & 1 & 0 & \rho_1 \\
0 & 0 & 1 & \rho_1 + \rho_2 \\
\end{pmatrix}
= \nu
\begin{pmatrix}
\kappa_1 & \kappa_2 & \kappa_1 + \kappa_2 & M \\
0 & 0 & 1 & \rho_1 + \rho_2 \\
0 & 0 & 1 & \rho_1 + \rho_2 \\
0 & 0 & 1 & \rho_1 + \rho_2 \\
\end{pmatrix}
$$

respectively, so the contributions of $T \cup \{v\}$ to $\langle G \rangle$ and $\langle G' \rangle$ are equal. Similarly, $T \cup \{w\}$ makes equal contributions to $\langle G \rangle$ and $\langle G' \rangle$, and so does $T \cup \{u,v,w\}$.

Also, the $GF(2)$-nullities of $\mathcal{A}(G)_{T \cup \{u,w\}}$ and $\mathcal{A}(G')_{T \cup \{v,w\}}$ are

$$
\nu
\begin{pmatrix}
\kappa_1 & \kappa_2 & \kappa_1 + \kappa_2 & M \\
0 & 1 & 1 & \rho_1 \\
1 & 0 & 1 & \rho_2 \\
1 & 1 & 1 & \rho_1 + \rho_2 \\
\end{pmatrix}
= \nu
\begin{pmatrix}
\kappa_1 & \kappa_2 & \kappa_1 + \kappa_2 & M \\
0 & 0 & 1 & \rho_1 \\
1 & 0 & 0 & \rho_2 \\
0 & 0 & 1 & \rho_1 + \rho_2 \\
\end{pmatrix}
= \nu
\begin{pmatrix}
\kappa_1 & \kappa_2 & \kappa_1 + \kappa_2 & M \\
0 & 0 & 1 & \rho_1 \\
0 & 1 & 0 & \rho_2 \\
1 & 0 & 0 & \rho_1 + \rho_2 \\
\end{pmatrix}
$$

and

$$
\nu
\begin{pmatrix}
\kappa_1 & \kappa_2 & \kappa_1 + \kappa_2 & M \\
1 & 0 & 1 & \rho_1 \\
0 & 1 & 0 & \rho_2 \\
0 & 0 & 1 & \rho_1 + \rho_2 \\
\end{pmatrix}
= \nu
\begin{pmatrix}
\kappa_1 & \kappa_2 & \kappa_1 + \kappa_2 & M \\
0 & 0 & 1 & \rho_1 \\
0 & 1 & 0 & \rho_2 \\
1 & 0 & 0 & \rho_1 + \rho_2 \\
\end{pmatrix}
= \nu
\begin{pmatrix}
\kappa_1 & \kappa_2 & \kappa_1 + \kappa_2 & M \\
0 & 0 & 1 & \rho_1 \\
0 & 1 & 0 & \rho_2 \\
1 & 0 & 0 & \rho_1 + \rho_2 \\
\end{pmatrix}
$$

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respectively, so the contributions of $T \cup \{u, w\}$ to $\langle G \rangle$ and $T \cup \{v, w\}$ to $\langle G' \rangle$ are equal. Similarly, the contributions of $T \cup \{u, w\}$ to $\langle G' \rangle$ and $T \cup \{v, w\}$ to $\langle G \rangle$ are equal. ■

**Definition 28** A marked-graph $\Omega$ Reidemeister move is obtained by composing one of the moves described in Proposition 27 with any finite sequence of marked pivots and $\Omega$ Reidemeister moves. The inverse of an $\Omega$ move is also an $\Omega$ move, as is the move obtained by first toggling all loops in the original graph, and then toggling all loops in the final graph.

Theorem 16, Proposition 27 and the results of Section 8 imply that $\Omega$ moves preserve the reduced marked-graph bracket.

Including composition with $\Omega$ moves in Definition 28 is suggested by Östlund’s observation that different-looking $\Omega$ moves are interrelated through composition with $\Omega$ moves [35]. The unmarked $\Omega$ moves used in [44] (see Figure 13) are interrelated in the same way, but composition with $\Omega$ moves was not useful in [44] because the proof of the invariance of the reduced graph bracket under $\Omega$ moves used induction on the number of vertices, and $\Omega$ moves may increase the number of vertices. Moreover the inductive argument used to verify invariance under one $\Omega$ move actually required invariance under different $\Omega$ moves on smaller graphs. The arguments we present here do not involve induction on the number of vertices, so Östlund’s observation is quite useful.

**Corollary 29** Suppose $G$ is a marked graph with three unmarked vertices $u, v, w$ such that the induced subgraph $G[\{u, v, w\}]$ is isomorphic to one of the six 3-vertex graphs pictured in Figure 13 and every vertex $x \notin \{u, v, w\}$ is adjacent to either 0 or precisely two of $u, v, w$. Then there is a marked-graph $\Omega$ Reidemeister move whose effect is to toggle all the non-loop edges among $u, v, w$.

**Proof.** The instance of the proposition involving the 3-vertex graph that appears at the top left of Figure 13 is verified in Proposition 27. As was observed in [44], the other two instances that appear in the top row of Figure 13 can be obtained from this one through composition with $\Omega$ moves. (The same constructions given in [44] apply here, because the $\Omega$ moves of Proposition 22 coincide locally with the unmarked $\Omega$ moves of [44], and do not require any special assumptions regarding the marked vertices of $G$.) The three instances pictured in the bottom row of Figure 13 follow, for if $G$ contains such an induced subgraph then we may toggle all loops in $G$, apply the corresponding move pictured in the top row of Figure 13, and then toggle all loops in the resulting graph. ■

N.B. Corollary 29 states only that the moves illustrated in Figure 13 are examples of marked-graph $\Omega$ moves. As Definition 28 includes composition
Figure 13: The examples of $\Omega_3$ moves mentioned in Corollary 29 involve toggling the non-loop edges in one of the six pictured configurations; every vertex outside the picture must have either 0 or precisely 2 neighbors among the three unmarked vertices that are pictured.

with $\Omega_2$ moves and marked pivots, it is not possible to give an exhaustive list of $\Omega_3$ moves.

If $D$ is a link diagram then an $\Omega_3$ Reidemeister move involves replacing a configuration like one of those in Figure 14 with a configuration like the other. The arcs appearing in the figure may be oriented in any way, and one may replace both configurations by their mirror images.

**Proposition 30** If $D'$ is obtained by applying an $\Omega_3$ Reidemeister move to a link diagram $D$ then there are Euler systems $C$ and $C'$ such that $\mathcal{L}(D', C')$ is obtained from $\mathcal{L}(D, C)$ by applying a marked-graph $\Omega_3$ move or a finite (possibly empty) sequence of marked pivots.

**Proof.** Let $C$ be an Euler system for the directed universe $\bar{U}$ of the diagram $D$ which contains the left-hand side of Figure 14 and let $D'$ be the link diagram that differs from $D$ only by containing the right-hand side of Figure 14.
Suppose first that the crossings are all unmarked.

If the horizontal arc in the left-hand portion of Figure 14 is oriented from left to right and the other two arcs are oriented from top to bottom, then the fact that the circuit of \(C\) that passes through this portion of \(D\) is an Euler circuit implies that it is either \(uvC_1uwC_2uwC_3\) or \(uvC_1vwC_2uwC_3\); the Euler circuit in the right-hand portion of Figure 14 that arises naturally from \(C\) will then be \(vuC_1wuC_2uwC_3\) or \(vuC_1vwC_2wuC_3\), respectively. All three crossings are positive, and each crossing outside the figure is interlaced with either 0 or precisely 2 of these; for instance a crossing that appears once in \(C_1\) and once in \(C_2\) is interlaced with \(u\) and \(w\) in the first case, and \(v\) and \(w\) in the second case. Moreover each crossing in the left-hand portion of the figure is interlaced with the same outside crossings as the crossing on the right that has the same label \((u, v \text{ or } w)\). It follows that the pictured \(\Omega\)-move is the one that appears in the middle of the top row of Figure 13.

If Figure 14 is replaced by its mirror image and the orientations are the same as in the preceding paragraph, the same argument holds with all loops toggled.

According to Lemma 3 of \[35\], every other \(\Omega\)-move represented by some oriented version of Figure 14 or its mirror image is a composition of \(\Omega\)-2 moves and one of the two moves just discussed. The result then follows from the two cases just discussed and Proposition 26.

Suppose now that the portion of \(D\) pictured on the left-hand side of Figure 14 has a single marked vertex \(x \in \{u, v, w\}\). If \(x\) is interlaced with any crossing \(y\) outside the pictured portion of \(D\), then we may replace \(C\) with \(C*xy*xy\), which has no marked vertices in the pictured portion of \(D\). It remains to consider the possibility that precisely one of \(u, v, w\) is marked and the marked element of \(\{u, v, w\}\) is not interlaced with any crossing outside \(\{u, v, w\}\). Once again there are several special cases, but we need only discuss one explicitly.
For instance, suppose that the left-hand side is oriented as in the first paragraph of the proof, \( v \) is marked and \( u, w \) are not. The circuit of \( C \) incident on the left-hand side of Figure 14 is \( uvwC_1vC_2uvwC_3 \) or \( vC_1uwCvC_3uwC_2v \); the obvious Euler system for the right-hand side of Figure 14 then includes the incident circuit \( vC_1uwvC_2uwC_3 \) or \( vC_2wuC_2uwv \), respectively. The assumption that \( v \) has no neighbor outside \( \{ u, w \} \) implies that no crossing appears precisely once within \( C_1 \); it follows that \( L(D', C') \) is actually identical to \( L(D, C) \). If only \( w \) is marked then the circuit of \( C \) incident on the left-hand side of Figure 14 is \( uwC_1vwC_2uwC_3 \) or \( vwC_1uwC_2uwC_3 \), and the obvious Euler system for the right-hand side of Figure 14 includes the incident circuit \( uvC_1wuC_2vuC_3 \) or \( uwC_1vuC_2wuC_3 \), respectively. The assumption that \( w \) has no neighbor outside \( \{ u, v \} \) implies that no crossing appears precisely once within \( C_1 \); it follows again that \( L(D', C') \) is identical to \( L(D, C) \). Finally, if \( u \) is the lone marked vertex in the left-hand portion of Figure 14 then the incident circuit of \( C \) is either \( uwC_1uvC_2uwv \) (in which \( u \) is interlaced with \( w \)) or \( uvC_1uwC_2vC_3 \) (in which \( u \) is interlaced with \( v \)). If \( u \) is interlaced with \( v \) then \( C \ast u \ast v \ast u \) will have \( v \) marked instead of \( u \), and if \( u \) is interlaced with \( w \) then \( C \ast u \ast w \ast u \) will have \( w \) marked instead of \( u \); either way a transposition leads to one of the first two cases.

As before, the mirror image is handled by toggling all loops, and other oriented versions of the left-hand side of Figure 14 (or its mirror image) with precisely one marked vertex are handled by expressing the corresponding moves as compositions of \( \Omega \) moves with one of the two \( \Omega \) moves already considered.

Suppose now that the portion of \( D \) pictured on the left-hand side of Figure 14 includes precisely two marked vertices. If a marked crossing \( x \in \{ u, v, w \} \) is interlaced with any crossing \( y \) outside the pictured portion of \( D \), then we may replace \( C \) with \( C \ast x \ast y \ast x \), which has only one marked vertex in the pictured portion of \( D \); hence we may assume no marked crossing among \( \{ u, v, w \} \) is interlaced with any crossing outside \( \{ u, v, w \} \). The hypothesis that every crossing outside \( \{ u, v, w \} \) is interlaced with an even number of elements of \( \{ u, v, w \} \) then allows us to assume that no crossing outside \( \{ u, v, w \} \) is interlaced with any of \( u, v, w \). Moreover, if the two marked crossings are interlaced then the marks are removed by replacing \( C \) with the Euler system obtained by a transposition on these two vertices, so we may proceed assuming they are not interlaced.

Suppose again that the left-hand side of Figure 14 is oriented as above. If \( u, v \) are the two crossings that are marked then the incident Euler circuit of \( C \) must be \( uvwC_1vuwC_2vC_3 \). There is then an Euler circuit \( vuC_1wuC_2wvC_3 \) incident on the right-hand portion of Figure 14 let \( C' \) be the Euler system that coincides with \( C \) except for this replacement of circuits incident on the pictured portions of \( D \) and \( D' \). The assumption that no outside crossing is interlaced with any of \( u, v, w \) with respect to \( C \) tells us that every outside crossing appears twice in one \( C_i \); hence no outside crossing is interlaced with any of \( u, v, w \) with respect to \( C' \) either. Considering \( uvwC_1uwvC_2vC_3 \), we see that \( L(D, C) \) has a connected component whose only vertices are \( u, v \) and \( w \); \( u \) and \( v \) are marked, nonadjacent neighbors of \( w \). Considering \( vuC_1wuC_2wvC_3 \), we see that \( L(D', C') \) has a connected component whose only vertices are \( u, v \) and \( w \); \( u \) and
If $u$ and $w$ are the two that are marked then the Euler circuit of $C$ incident on the left-hand portion of Figure 14 must be $uvC_1uwC_2vwC_3$; there is then an Euler circuit $wvuC_1vC_2wuC_3$ incident on the right-hand portion. If $C'$ is the Euler system that differs from $C$ only by including this latter circuit then $L(D', C') = L(D, C)_{vw}$. Finally, if $v$ and $w$ are the two that are marked then the Euler circuit of $C$ incident on the left-hand portion of Figure 14 must be $uwC_1vC_2uwC_3$; hence an Euler circuit incident on the right-hand portion is $wvC_1vuC_2vC_3$. If $C'$ is the Euler system that differs from $C$ only by including this latter circuit then $L(D', C') = L(D, C)_{vw}$.

Finally, suppose all three of $u, v, w$ are marked with respect to $C$. Any adjacency involving two of them allows a marked pivot that will eliminate two of the three marks, so we may assume no two of $u, v, w$ are interlaced with respect to $C$. This is not possible.

**Corollary 31** Suppose $D$ and $D'$ are diagrams of the same oriented virtual link type, with directed universes $\vec{U}$ and $\vec{U}'$. Let $C$ and $C'$ be directed Euler systems for $\vec{U}$ and $\vec{U}'$. Then $L(D, C)$ and $L(D', C')$ can be transformed into each other through some finite sequence of marked pivots, marked-graph $\Omega_2$ and $\Omega_3$ Reidemeister moves, and adjunctions and deletions of isolated, unmarked vertices.

**Proof.** $D$ and $D'$ can be transformed into each other through some finite sequence of Reidemeister moves of types $\Omega.1$, $\Omega.2$ and $\Omega.3$ and virtual Reidemeister moves. $\Omega.1$ moves correspond to adjunctions and deletions of isolated, unmarked vertices. Virtual Reidemeister moves have no effect on $L(D, C)$. $lacksquare$

**10 The Jones polynomial**

**Definition 32** A Reidemeister move on a marked graph is one of the following.

$\Omega.1$. An isolated, unmarked vertex may be adjoined or deleted. The vertex may be looped or unlooped.

$\Omega.2$. One of the marked-graph $\Omega.2$ Reidemeister moves of Section 8 may be applied.

$\Omega.3$. One of the marked-graph $\Omega.3$ Reidemeister moves of Section 9 may be applied.

**Definition 33** Let $G$ be a marked graph with $n$ vertices, $\ell$ of which are looped. Then the Jones polynomial of $G$ is

$$V_G(t) = (-1)^n \cdot t^{(3n-6\ell)/4} \cdot \langle G \rangle (t^{-1/4}).$$

If $G$ has no free loops and no marked vertices then the bracket, reduced bracket and Jones polynomial of $G$ coincide with those discussed in [44]. Much of the discussion of [44] is therefore still relevant. In particular, several important aspects of the classical theory of the Jones polynomial do not extend to
this context; for instance, we cannot expect to calculate \( V_G(t) \) using a recursive algorithm that reduces \( G \) to “ungraphs” in the same way that the Jones polynomial of a classical link can be calculated by repeatedly using Theorem 12 of [17] (the correct version of which is the formula \( t^{-1}V_{L^+} - tV_{L^-} = (t^{1/2} - t^{-1/2})V_L \)) to reduce \( L \) to unlinks.

However the fundamental property that the Jones polynomial is a link type invariant does survive the generalization to marked graphs.

**Theorem 34** If \( G \) is a marked graph and \( G' \) is obtained from \( G \) through marked pivots and Reidemeister moves then \( V_G = V_{G'} \).

**Proof.** A marked pivot or \( \Omega \).3 move does not affect \( n, \ell \) or \( \langle G \rangle \).

If \( E_1 \) consists of one isolated, unmarked, unlooped vertex then the disjoint union \( G \cup E_1 \) has \( |G \cup E_1| = (Ad + B) \cdot |G| \), so \( \langle G \cup E_1 \rangle = (-A^3) \cdot \langle G \rangle \) and \( V_{G \cup E_1} = -(−1)^n t^{3/4} \cdot t^{(3n−6\ell)/4} \cdot (−t^{−3/4}) \langle G \rangle \cdot (t^{−1/4}) = V_G \), where \( n \) and \( \ell \) refer to \( G \). A similar calculation shows that adjoining a single isolated, unmarked, looped vertex does not change \( V \).

An \( \Omega \).2 move has no effect on \( \langle G \rangle \). It adds 2 or −2 to the number of vertices while adding 1 or −1 (resp.) to the number of loops, so it also has no effect on \( (−1)^n \cdot t^{(3n−6\ell)/4} \).

In closing we observe that a special feature of the marked-graph bracket is that both definitions (the nullity formula and the recursion) are “local” in the sense that information from different connected components of \( G \) is processed separately. Consequently there are multivariable forms of the marked-graph bracket and Jones polynomials, in which different connected components of \( G \) are associated with different sets of variables. For classical knots, disconnected interlacement graphs arise from connected sums, which are well understood [39]; on the other hand, the connected sum of virtual knot types is not uniquely defined [23, 24]. Whether there is any useful application of the multivariable bracket and Jones polynomials remains to be seen.

**References**


