Design theory and algebraic coding theory have their origins in disparate fields of study: the statistical theory of the design of experiments and the theory of information transmission in electrical engineering. Yet each has enriched the other by providing tools capable of answering interesting and fundamental questions. A primary topic of this exposition will be the study of biplanes and projective planes, focusing on certain interesting relationships between these two types of combinatorial designs, relationships which are uncovered by exploiting the techniques of coding theory.

We shall set the stage in Section 1 by presenting a fairly complete introduction to the theory of designs, thereby putting both planes and biplanes in proper perspective. Section 2 presents the terminology of coding theory and a few of its tools. Then we examine both biplanes and planes from the viewpoint of coding theory. Additional design-theoretic tools are developed in Section 3, and in Section 4 much of the preceding is brought to bear in elucidating an elegant coding-theoretic link between planes and biplanes.

1. Introduction to Design Theory. A $t$-design (or $t-(v, k, \lambda)$ design) on a $v$-set $S$ (a finite set of size $v$ whose elements are called points) is a collection, $\mathcal{B}$, of $k$-subsets of $S$ (called blocks) such that every $t$-subset of $S$ is contained in precisely $\lambda$ elements of $\mathcal{B}$. We note that, although there are infinitely many nontrivial $t$-designs for each $t < 5$, no nontrivial designs are known for $t > 5$.

The collection consisting of all $k$-subsets of a $v$-set $S$ trivially forms a complete $k$-$(v, k, 1)$ design. The interesting case is when the collection is incomplete in the sense that not all $k$-subsets are present. In the statistical theory of the design of experiments, a primary root of design theory [12], [57], an incomplete 2-design is called a balanced incomplete block design (BIBD). The term “balanced” refers to the property that each pair of points, or treatments, as they are called, is contained in exactly $\lambda$ blocks.

Below are three examples of (incomplete) $t$-designs. We shall often have occasion to refer to them. Each is related to the triangular figure presented in Fig. 1.

Example 1.1. In this figure, we have seven points and seven geometrical objects (six line segments and one circle), each of which is incident with three points. Let us call these seven objects “lines.” A $2-(7, 3, 1)$ design is obtained by letting $S$ consist of these seven points and letting a block consist of the three points incident with one of the seven lines. So, $\mathcal{B} = \{B_i|1 \leq i < 7\}$, where

\[
\begin{align*}
B_1 &= \{1, 2, 4\} & B_2 &= \{2, 3, 5\} & B_3 &= \{3, 4, 6\} \\
B_4 &= \{4, 5, 7\} & B_5 &= \{5, 6, 1\} & B_6 &= \{6, 7, 2\} \\
B_7 &= \{7, 1, 3\},
\end{align*}
\]
and one can quickly check that every pair of points is contained in precisely one block.

**Example 1.2.** We can also obtain a 2-(7, 4, 2) design from this figure. Again, let $S$ be the set of seven points. Let a block consist of the four points not incident with one of the seven lines. Thus

$$
B_1 = \{3, 5, 6, 7\} \quad B_2 = \{1, 4, 6, 7\} \quad B_3 = \{1, 2, 5, 7\}
$$

$$
B_4 = \{1, 2, 3, 6\} \quad B_5 = \{2, 3, 4, 7\} \quad B_6 = \{1, 3, 4, 5\}
$$

$$
B_7 = \{2, 4, 5, 6\}.
$$

Now, each pair of points is contained in precisely two blocks.

In each of the previous two examples, the number of blocks is the same as the number of points, each point is incident with the same number of blocks, this number being $k$, the number of points incident with a given block, and any two blocks meet in precisely $\lambda$ points. Are these properties true in general? Let us examine the next example.

**Example 1.3.** Let $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and let $\mathcal{D}$ consist of the seven blocks of Example 1.2 and the seven blocks obtained by adjoining point "8" to each of the seven blocks of Example 1.1. One can verify that each 3-subset is contained in precisely one block, and so this collection of fourteen blocks forms a 3-(8, 4, 1) design on the set $S$.

In this last example, each point is incident with the same number of blocks, namely seven, but the other "properties" fail. Also, each pair of points is incident with precisely three blocks. Hence this 3-design is a 1-(8, 4, 7) and a 2-(8, 4, 3) design as well.

**Proposition 1.4.** If $(S, \mathcal{D})$ is a $t$-$(v, k, \lambda)$ design, then it is also a $\tau$-$(v, k, \lambda, \tau)$ design for $0 < \tau < t$, with

$$
\lambda \binom{k - \tau}{t - \tau} = \lambda \binom{v - \tau}{t - \tau}.
$$

**Proof.** Let $X_t$ denote a set of $t$ points. Let $\bar{X}_\tau$ be a given set of $\tau$ points and let $\bar{X}_\tau$ be the number of blocks containing $\bar{X}_\tau$. Now, $\bar{X}_\tau$ is contained in $\bar{X}_\tau$ blocks, each containing $k - \tau$ points other than those of $\bar{X}_\tau$, and so each of these $\bar{X}_\tau$ blocks yields $\binom{k - \tau}{t - \tau}$ ways of extending $\bar{X}_\tau$ to a $t$-subset. Similarly, $\bar{X}_\tau$ can be extended to a $t$-subset in $\binom{v - \tau}{t - \tau}$ ways, and each resulting $t$-subset is contained in $\lambda$ blocks. Thus, by counting the number of ordered pairs in the set

$$\{(X_{t - \tau}, B) | B \in \mathcal{D}, \bar{X}_\tau \cup X_{t - \tau} \subset B\}$$
in two ways, we obtain
\[
\binom{k-\tau}{t-\tau}\lambda = \binom{v-\tau}{t-\tau}\lambda.
\]
Solving for \(\bar{\lambda}_v\), we see it depends only on the parameters of the design, and so \(\bar{\lambda}_v\) is independent of the particular \(\bar{X}_v\) chosen. Therefore \((S, \mathcal{B})\) is a \(\tau\)-design.

Thus for a \(t-(v, k, \lambda)\) design to exist, it is necessary that \(\lambda \binom{v-\tau}{t-\tau}/ \binom{k-\tau}{t-\tau}\) be an integer for \(0 < \tau < t\). This condition is not sufficient, however, since, for example, there does not exist a 2-(43, 7, 1) design, despite the fact that the numerical condition is satisfied for \(\tau = 0, 1,\) and 2.

Let us denote \(\lambda_0\), the number of blocks, by \(b\), and \(\lambda_1\), the number of blocks containing any given point, by \(r\). Setting \(\tau = 0\) in Proposition 1.4 we obtain
\[
b\left(\binom{k}{t}\right) = \lambda\left(\binom{v}{t}\right).
\]
So, in any 1-design we have
\[
bk = rv,
\]
since \(\lambda = \lambda_1 = r\). We can obtain this last identity directly by counting the number of ordered pairs in the set
\[
\{(P, B) | P \in S, B \in \mathcal{B}, P \in B\}
\]
in two ways. Also, in any 2-design we have, by taking \(\tau = 1\) in Proposition 1.4,
\[
r(k - 1) = \lambda(v - 1).
\]

The order of a 2-design is \(n = r - \lambda\). As we shall see, this parameter plays an important role in design theory.

The 3-(8, 4, 1) design of Example 1.3 is also a 2-(8, 4, 3) design, a 1-(8, 4, 7) design \((r = 7)\), and a 0-(8, 4, 14) design \((b = 14)\). Its order \(n = 7 - 3 = 4\).

In addition to trying to determine, for a given set of parameters, whether a design exists or not; that is, whether a \(t-(v, k, \lambda)\) design exists for a particular choice of \(t, v, k,\) and \(\lambda\), it is also of importance to enumerate, if possible, all designs with a given set of parameters, once one has determined that at least one such design exists. We say two designs are equivalent or isomorphic if there exists a bijection of their point sets which transforms the blocks of one into those of the other. That is, two designs are equivalent if one can be obtained from the other simply by renaming the points and blocks, and one wishes, of course, to enumerate the isomorphism classes. The automorphism group of the design is
\[
\{\sigma \in \text{Sym}(S) | \sigma \cdot B \in \mathcal{B}, \text{for all } B \in \mathcal{B}\},
\]
where \(\sigma \cdot B = \{\sigma(P) | P \in B\}\); that is, all permutations which leave the block structure invariant.

We next develop some tools and techniques which will help us to deal with the questions of existence and equivalence of designs.

The incidence matrix \(M\) of a design is a \(b \times v\) matrix \((m_{B,P})\) where \(m_{B,P} = 1\) if \(P \in B\) and 0 otherwise. In the following, let \(I\) denote the identity matrix and \(J\) the matrix (of the appropriate size) whose every entry is 1.

**Lemma 1.5.** In any 2-design:

1. \(MJ = kJ\).
2. \(JM = rJ\).
3. \(M' M = (r - \lambda)I + \lambda J = nI + \lambda J\).
4. \(\det(M' M) = r k (r - \lambda)^{v - 1} = r k n^{v - 1}\).

**Proof.** (1) and (2) are obtained directly from the definitions of \(M\) and \(J\).
(3) A column of $M$ contains $r$ 1's and two different columns have $\lambda$ 1's in common.

(4) \[
M'M = \begin{bmatrix}
 r & \lambda & \cdots & \lambda \\
 \lambda & r & \cdots & \lambda \\
 \vdots & \vdots & \ddots & \vdots \\
 \lambda & \lambda & \cdots & r 
\end{bmatrix}
\]

Subtract the first column from each other column and obtain:

\[
\begin{bmatrix}
 r - n & -n \\
 \lambda & n \\
 \vdots & \vdots \\
 \lambda & 0 & n 
\end{bmatrix}
\]

Add each row to the first row and obtain:

\[
\begin{bmatrix}
 r+\lambda(v-1) & 0 \\
 \lambda & n \\
 \vdots & \vdots \\
 \lambda & 0 & n 
\end{bmatrix}
\]

Therefore $\det(M'M) = (r+\lambda(v-1))n^{v-1} = rkn^{v-1}$, since $r(k-1) = \lambda(v-1)$.

In each of our three examples there were at least as many blocks as there were points. Our next proposition, known as Fisher's Inequality, establishes this property for all interesting designs.

**Proposition 1.6.** If $(S, \mathcal{B})$ is a $t$-design with $t > 2$ and $k < v - 1$, then $b > v$.

**Proof.** $(S, \mathcal{B})$ is a 2-design. Recall $r(k-1) = \lambda(v-1)$. So, if $r = \lambda$, then $v = k$. Therefore $\det(M'M) \neq 0$, and so $M'M$ is nonsingular, and hence of rank $v$. Thus $v = \text{Rk}(M) < b$, since $M$ is a $b \times v$ matrix.

The hypothesis of a 2-design in Proposition 1.6 cannot be weakened, since if we let $S = \{1, 2, 3, 4\}$ and $\mathcal{B} = \{(1, 2), (3, 4)\}$, then $(S, \mathcal{B})$ is a 1-(4, 2, 1) design. But $b = 2 < v = 4$.

Let $(S, \mathcal{B})$ be a $t$-(v, k, \lambda) design. Pick $P \in S$. Let $S_P = S - \{P\}$ and $\mathcal{B}_P = \{B - \{P\} \mid B \in \mathcal{B}, P \in B\}$. $(S_P, \mathcal{B}_P)$ is called the contraction of $(S, \mathcal{B})$ at $P$ and it is a $(t-1)$-(v-1, k-1, \lambda) design. Note that the "r" for $(S, \mathcal{B})$ is the "b" for $(S_P, \mathcal{B}_P)$.

Thus the existence of a single $t$-design establishes the existence of several possibly nonisomorphic $(t-1)$-designs. In Example 1.3, by contracting on point "8", we obtain the design of Example 1.1. By contracting on any other point, we obtain a design isomorphic to that of Example 1.1.

We immediately exploit this notion of contraction in proving the next proposition.

**Proposition 1.7.** If $(S, \mathcal{B})$ is an incomplete $t$-(v, k, \lambda) design with $t > 3$, then $b > v$.

**Proof.** We know $b > v$ by Proposition 1.6. By contracting on a point $P$ of $S$, we obtain a $(t-1)$-(v-1, k-1, \lambda) design. Since $t - 1 > 2$, we have, by Proposition 1.6, $r = b_{S_P} > v - 1$, where $b_{S_P}$ denotes the number of blocks in the $(t-1)$-design. Say $b = v$. Then, since $vr = bk$, we get $k = r = v - 1$. If $k = v$, we have the complete $v$-(v, 1) design, and if $k = v - 1$, we have the complete $(v - 1)$-(v, v-1, 1) design. Thus $b > v$. 
There is a special name for the interesting and important class of 2-designs for which the number of blocks and points is the same. A projective (or symmetric) design is one for which \( t > 1 \) and \( b = v \) (and so \( r = k \) as well). Thus, except for some complete designs, a projective design is a 2-design, and no more; that is, it cannot be a \( t \)-design with \( t > 3 \). We shall therefore suppress the "2" and speak of projective \((v, k, \lambda)\) designs.

The dual of a design \((S, \mathfrak{D})\), denoted by \((S^d, \mathfrak{D}^d)\), is obtained by interchanging the notions of point and block. If \((S, \mathfrak{D})\) is a \(1-(v, k, r)\) design, then \((S^d, \mathfrak{D}^d)\) is a \(1-(b, r, k)\) design. Can both the design and its dual be better than 1-designs? If the design is no better than a 1-design, then clearly no. If it is an incomplete 3-design, then Proposition 1.7 implies that the dual design will have fewer blocks than points, and so Fisher's Inequality (Proposition 1.6) provides us with the negative answer. The next proposition provides us with an affirmative answer in the case of the projective design.

**Proposition 1.8.** If \((S, \mathfrak{D})\) is a projective \((v, k, \lambda)\) design, then \((S^d, \mathfrak{D}^d)\) is as well; that is, any two distinct blocks of \(\mathfrak{D}\) have precisely \(\lambda\) points in common.

**Proof.** As we saw in the proof of Fisher's Inequality, \(v \leq \text{Rk}(M) \leq b\). Since \(b = v\) for a projective design, \(\text{Rk}(M) = v\). So \(M^{-1}\) exists and by (3) of Lemma 1.5 we obtain \(M' = [(r-\lambda)I + \lambda J]M^{-1}\). Since \(r = k\), parts (1) and (2) of Lemma 1.5 imply \(MJ = JM\). Therefore \(MM' = M'M = (r-\lambda)I + \lambda J\). Hence any two distinct blocks of \(\mathfrak{D}\) have precisely \(\lambda\) points in common.

In the remainder of this exposition we shall be concerned primarily with projective designs for which \(\lambda = 1\) or 2. A (projective) plane is a projective design with \(\lambda = 1\); that is, an \((n^2 + n + 1, n + 1, 1)\) design, where \(n = k - 1\) is the order. The blocks of a plane are commonly called lines. A biplane is a projective design with \(\lambda = 2\); that is, a \(\frac{1}{2}(n^2 + 3n + 4), n + 2, 2\) design, where \(n = k - 2\) is the order. The parameters of these projective designs are obtained by using the identity \(k(k-1) = \lambda(v-1)\), solving for \(v\), and then expressing each parameter in terms of the order.

The design of Example 1.1 is a plane of order 2 and the design of Example 1.2 is a biplane of order 2.

Only finitely many projective designs are known to exist for any fixed \(\lambda > 1\), and it is conjectured that indeed there exist but finitely many projective designs for any fixed \(\lambda > 1\). However, an infinite class of planes, called Desarguesian planes of order \(n\), are constructed from finite fields with \(n = p^a\) elements where \(p\) is a prime and \(a\) is a positive integer [23], [29], [53]. We shall indicate a construction in Section 2.

We shall not call the complete 2-(3,2,1) design a plane of order 1. Normally, one of the defining properties of a projective plane is that it contain a set of four triply noncollinear points. Each of our planes will satisfy this property.

The following chart exhibits the current state of affairs for biplanes.

<table>
<thead>
<tr>
<th>(n)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>number</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>&gt; 4</td>
<td>0</td>
<td>&gt; 2</td>
</tr>
</tbody>
</table>

It indicates that the biplanes of orders 1 through 3 are unique. Also, there exist precisely three biplanes of order 4 [33], four of order 7 [49], at least four of order 9 [50], and at least two of order 11 [2]. The two of order 11 and two of the biplanes of order 7 are duals of each other and the rest are self-dual (they are isomorphic to their duals). We shall discuss many of these biplanes in more detail in the succeeding sections.

The following theorem presents the strongest known necessary condition for the existence of a projective design. It is the fundamental theorem of the subject. It was first proved by Bruck and Ryser [14] for the case of planes (see Corollary), and then extended by Chowla and Ryser [17] to cover an arbitrary projective design.
**Theorem 1.9.** If \((S, \mathcal{D})\) is a projective \((v, k, \lambda)\) design of order \(n=k-\lambda\), then

1. \(v\) even implies \(n\) is a square,
2. \(v\) odd implies \(x^2 = ny^2 + (-1)^{(1/2)(v-1)}\lambda z^2\) has a solution in integers \(x, y,\) and \(z\), not all 0.

**Proof.** By Lemma 1.5, \(\det(M) = \det(M') = (\det(M'M))^{1/2} = (rk(r-\lambda)^{v-1})^{1/2} = kn^{(1/2)(v-1)}\). Hence, if \(v\) is even, then \(n^{1/2}\) must be an integer.

The proof of (2) involves sophisticated number theoretic techniques, and we shall not present it here.

It is conceivable that these necessary conditions are even sufficient, since the nonexistence of a design with parameters satisfying these conditions has never been proved.

Since a biplane of order 5 would be a \((22,7,2)\) projective design, part (1) of the theorem guarantees its nonexistence.

**Corollary.** If \(n \equiv 1\) or \(2\) (mod 4) and \(n\) cannot be expressed as the sum of two integral squares, then there does not exist a plane of order \(n\).

Since \(6 \equiv 2\) (mod 4) and \(6\) cannot be expressed as the sum of two integral squares, there does not exist a plane of order 6.

It is possible to prove by means of algebraic coding theoretic techniques that there exists no plane of order \(n \equiv 6\) (mod 8) and no biplane of order \(n \equiv 5\) (mod 8) [3].

There have been many attempts to construct the putative plane of order 10 (see [40]), which is the smallest unsettled case, and we shall present one such attempt in Section 4. Such a plane would, of course, be a projective \((111, 11, 1)\) design, and the search space is simply much too large to exhaust all possibilities by electronic computation on a computer. It is conjectured by some, however, that every plane is of prime power order. Certainly every known plane is of prime power order.

In addition to the existence question, the question of enumeration is also of interest; that is: How many nonisomorphic planes exist for a given order? It is known that there exist planes other than those constructed from the finite fields. There exist unique planes of orders 2, 3, 4, 5, 7, and 8 (see [28]), but there exist at least four nonisomorphic planes of order 9 (see [27] and [53, p.72]), and at least five nonisomorphic planes of order 16 [36], [37]. Of these, the non-Desarguesian planes are constructed from Veblen-Webberburn systems (quasifields) and semifields (nonassociative division rings).

Inverse to the operation of contracting a design, which we dealt with previously, is the notion of possibly extending a design. A \((r+1)\)-design \((S, \mathcal{D})\) is said to be an extension of a \(t\)-design \((S', \mathcal{D}')\) if \((S', \mathcal{D}')\) is isomorphic to \((S_{\mathcal{D}^*}, \mathcal{D}_{\mathcal{D}^*})\), the contraction of \((S, \mathcal{D})\) at \(P\), for some point \(P\) in \(S\).

The 3-(8,4,1) design of Example 1.3 is the unique extension of the plane of order 2, the 2-(7,3,1) design of Example 1.1. This 3-design cannot be extended further (see Lemma 1.10 below).

**Lemma 1.10.** If a \(t-(v, k, \lambda)\) design is extendible, then \(k+1\) divides \(b(v+1)\).

**Proof.** Denote a parameter of the extension by placing a \(^{-}\) above the letter. The extension, if it exists, is a \(t-(\bar{v}, \bar{k}, \bar{\lambda})=(t+1)-(v+1, k+1, \lambda)\) design with \(\bar{r}=\bar{b}\). Thus \(\bar{b}\bar{k}=\bar{v}\bar{r}\) implies \(\bar{b}(k+1)=(v+1)b\).

**Theorem 1.11.** If a plane of order \(n\) is extendible, then \(n=2, 4,\) or 10 (see [31]).

**Proof.** A plane of order \(n\) is a 2-(\(n^2 + n + 1\), \(n + 1, 1\)) design and its extension would be a 3-(\(n^2 + n + 2\), \(n + 2, 1\)) design. By Lemma 1.10, \(b(v+1) \equiv 0\) (mod \(k+1\)). That is,

\[(n^2 + n + 1)(n^2 + n + 2) \equiv 0\pmod{n + 2}\]

Since \(n \equiv -2\) (mod \(n + 2\), we obtain
Thus $n = 2, 4,$ or $10.$

Thus if the plane of order $10$ does exist, it might have an extension. The plane of order $4,$ a $2-(21,5,1)$ design, can be uniquely extended three times to a $5-(24,8,1)$ design. By Lemma 1.10, however, it can be extended no further.

We conclude this section with a method of constructing infinitely many projective designs with $\lambda > 1$ via the planes (of course, but finitely many for each fixed $\lambda$). The next lemma (see [1] and [41]) prepares the way.

**Lemma 1.12.** For a $t-(v,k,\lambda)$ design, the number of blocks meeting an $i$-subset of $S$ in precisely $j$ points, where $j < i < t,$ depends only on $i, j,$ and the parameters of the design.

**Proof.** Let $N_j$ equal the number of blocks meeting a given $i$-subset, say $X_i,$ in exactly $j$ points. Then we obtain the following system of $i + 1$ equations in $i + 1$ unknowns:

\[
N_0 + N_1 + N_2 + \cdots + N_i = \lambda_0 = b \\
N_1 + 2N_2 + 3N_3 + \cdots + iN_i = i\lambda_1 = ir \\
N_2 + \left(\frac{3}{2}\right)N_3 + \cdots + \left(\frac{i}{2}\right)N_i = \left(\frac{i}{2}\right)\lambda_2 \\
\vdots \\
N_i = \left(\frac{i}{i}\right)\lambda_i.
\]

Recall that, by virtue of Proposition 1.4, a $t$-design is also a $\tau$-design for $0 < \tau < t.$ The second equation is obtained by counting the number of ordered pairs in the set \[
\{(P,B)|P \in X_i, P \in B\}
\]
in two ways. The third is obtained similarly by counting the elements of \[
\{(P_1,P_2,B)|\{P_1,P_2\} \subseteq X_i, \{P_1,P_2\} \subseteq B\}
\]
in two ways. And so forth. This system can be solved simultaneously by starting with the last equation and proceeding to the first, and the resulting $N_j$'s are therefore seen to be independent of the particular $X_i$ chosen.

**Proposition 1.13.** Let $(S, \mathcal{D})$ be a projective $(v,k,\lambda)$ design, and let $\mathcal{D}^c = \{S - B | B \in \mathcal{D}\};$ then $(S, \mathcal{D}^c),$ called the complement of $(S, \mathcal{D}),$ is a projective

\[(v, v-k,(v-k)(v-k-1)/(v-1)) \text{ design}.
\]

**Proof.** If it is a 2-design, it will clearly be a projective design since the number of blocks and points is the same as for the original design. Now, it is a 2-design since the number of blocks of $(S, \mathcal{D})$ not meeting a 2-subset of $S$ is $\lambda_c = N_0 = v - N_1 - N_2$ (by Lemma 1.12 with $i = t = 2$), where $\lambda_c$ is the "$\lambda$" for the complementary design. Using the equations derived in the proof of Lemma 1.12 and solving for $N_2, N_1,$ and $N_0$ we obtain:

\[
N_2 = \lambda, \\
N_1 = 2k - 2N_2 = 2k - 2\lambda,
\]
and so
\[\lambda_c = N_0 = v + \lambda - 2k \]
\[= v + \frac{k(k - 1)}{v - 1} - 2k, \text{ since } \lambda\left(\frac{v}{2}\right) = b\left(\frac{k}{2}\right),\]
\[= (v - k)(v - k - 1)/(v - 1).\]

Thus the parameters of the design are correct.

The complementary design of the plane of Example 1.1 is the biplane of Example 1.2. In this manner, we can construct infinitely many projective designs with \(\lambda > 1\) as the complements of projective planes.

The reader may wish to consult Biggs [10], Cameron and van Lint [16], Dembowski [18], Hall [24], and Ryser [45] for additional material on design theory.

2. Codes and Ranks of Incidence Matrices. Coding theory has its roots in the problem of transmitting and correctly recovering information symbols which are sent over a noisy channel (see [9]). Surprisingly, there is great interplay between coding theory and design theory. We shall exploit this interrelationship and use certain tools of coding theory to answer some questions in the theory of designs in this and later sections. We first present some basic notions of algebraic coding theory.

A linear \((n,l)\) code \(A\) over the field \(F_q\) with \(q\) elements, \(q\) a prime power, is an \(l\)-dimensional subspace of \(F_q^n\), the vector space of all \(n\)-tuples with entries from \(F_q\). The elements of \(A\) are called code words or vectors. For \(a \in A\), the weight of \(a\) is
\[\text{wgt}(a) = |\{i|a_i \neq 0\}|,\]
where \(a = (a_1,a_2,\ldots,a_n)\), and \(|X|\) denotes the number of elements in the set \(X\). The minimum weight of \(A\) is
\[d(A) = \text{Min}\{\text{wgt}(a)|a \in A, a \neq 0\}.\]

The orthogonal of \(A\) is
\[A^\perp = \{b \in F_q^n|a \cdot b = 0, \forall a \in A\},\]
where \(a \cdot b\) is the usual dot product. If \(A\) is an \((n,l)\) code over \(F_q\), then \(A^\perp\), its orthogonal, is an \((n,n-l)\) code over \(F_q\) (see [39, p. 26]). Note that since the arithmetic is performed in \(F_q\), the dot product of a nonzero vector with itself might very well be zero. So we shall say \(A\) is self-orthogonal if \(A \subset A^\perp\) and self-dual if \(A = A^\perp\).

If one defines the distance between two code vectors \(a\) and \(b\) of \(A\) by \(\text{dist}(a,b) = \text{wgt}(a - b)\), then this distance function is a metric for the code \(A\).

For additional information on the subject of codes, see Assmus and Mattson [4], Berlekamp [9], Cameron and van Lint [16], MacWilliams and Sloane [39], Peterson and Weldon [43], and van Lint [55].

Let \(M\) be the incidence matrix of a 2-design \((S, \mathbb{D})\) and let \(p\) be a prime. The \(p\)-rank of \(M\), denoted by \(\text{Rk}_p(M)\), is the dimension over \(F_p\) of the rowspace of \(M\); that is, the rows of \(M\) generate a \((v, \text{Rk}_p(M))\) code over \(F_p\), which we shall denote by \(A_p\) or \(SP_p(M)\). For \(a \in A_p\), the support of \(a\) is defined to be
\[\text{supp}(a) = \{P_i|P_i \in S, a_i \neq 0\}.\]

The next theorem, primarily due to Hamada [29], is one illustration of the importance of the parameter \(n\), the order of a design.

Let us denote \((1,1,\ldots,1)\), the all-one vector, by \(1\).

**Theorem 2.1.** Let \(M\) be the incidence matrix of a \((v,k,\lambda)\) design, then \(\text{Rk}_p(M)\) depends only on the parameters of the design unless \(p\) divides \(n\), the order (in which case \(\text{Rk}_p(M)\) may depend on \(p\)).
upon the block structure of the design). In particular:

1. If \( p \mid r n \), then \( \text{Rk}_p(M) = \nu \).
2. If \( p \mid r \) and \( p \mid kn \), then \( \text{Rk}_p(M) = \nu \).
3. If \( p \mid r, p \mid k \), and \( p \mid n \), then \( \text{Rk}_p(M) = \nu - 1 \) and \( A_p = \langle 1 \rangle ^\perp \), the orthogonal of the code generated by the all-one vector.

**Proof.** Recall \( n = r - \lambda \).

1. Let \( w \) be the vector sum of all the rows of \( M \); then \( w = (r, r, \ldots, r) \). Denote by \( s_i \) the vector sum of those rows of \( M \) which have a 1 in column \( i \); that is, \( s_i \) is constructed from the blocks containing point \( P_i \). Then \( s_i = (\lambda, \lambda, \lambda, r, \lambda, \ldots, \lambda) \), where \( r \) inhabits the \( i \)th position. Thus
   \[
   rs_i - \lambda w = (0, \ldots, 0, r n, 0, \ldots, 0) \equiv 0 \pmod{p},
   \]
thereby yielding a collection of \( \nu \) linearly independent vectors. Hence \( \text{Rk}_p(M) = \nu \).

2. With \( s_i \) as before, consider
   \[
s_i - s_e = (0, \ldots, 0, n, 0, \ldots, 0, -n) \equiv 0 \pmod{p},
   \]
for \( 1 \leq i \leq \nu - 1 \). Each \( s_i - s_e \) is in \( \langle 1 \rangle ^\perp \), which is \( (\nu - 1) \)-dimensional, and \( \{s_i - s_e | 1 \leq i \leq \nu - 1\} \) is linearly independent. Hence the code spanned by \( \{s_i - s_e | 1 \leq i \leq \nu - 1\} \) equals \( \langle 1 \rangle ^\perp \). Thus \( \langle 1 \rangle ^\perp \) is a subspace of \( A_p \) and \( \text{Rk}_p(M) = \nu - 1 \). Now each row of \( M \) is necessarily in \( A_p \), but cannot be in \( \langle 1 \rangle ^\perp \) since \( p \nmid k \). Therefore \( \text{Rk}_p(M) = \nu \).

3. From part (2), we saw that \( \text{Rk}_p(M) = \nu - 1 \). If \( p \mid k \), then \( A_p \subseteq \langle 1 \rangle ^\perp \), since each row of \( M \) has \( k \)'s, and so \( \text{Rk}_p(M) < \nu - 1 \). Hence \( \text{Rk}_p(M) = \nu - 1 \) and \( A_p = \langle 1 \rangle ^\perp \).

Thus only when \( p \) divides the order is the notion of \( p \)-rank possibly useful in differentiating among designs with the same parameters. (If the incidence matrices of two designs have different \( p \)-ranks, then they are necessarily nonisomorphic.) For the three biplanes of order 4, only the 2-rank is of interest in possibly distinguishing them. In fact, the 2-ranks of the three are 6, 7, and 8, respectively.

In the case of a projective design, we can say a little more about the situation in which the prime of interest divides the order of the design.

**Proposition 2.2.** Let \( M \) be the incidence matrix of a projective \( (v, k, \lambda) \) design. If \( p \mid n \) but \( p^2 \nmid nk \), then \( \text{Rk}_p(M) \geq \frac{v}{2}(v + 1) \).

**Proof.** By the theory of elementary divisors, there exist unimodular matrices (matrices with integral coefficients and with a determinant of 1) \( P \) and \( Q \) such that \( PMQ \) is the diagonal matrix \( \{d_1, d_2, \ldots, d_e\} \), where \( d_1 | d_2 | \cdots | d_e \). The product of the elementary divisors is \( d_1 d_2 \cdots d_e = \det(PMQ) = \det(M) = kn^{(1/2)(v-1)} \) (by part (4) of Lemma 1.5 with \( r = k \)). Thus \( p \) divides \( \det(M) \) a total of \( \frac{1}{2}(v - 1) \) times. Hence \( \text{Rk}_p(M) = \text{Rk}_p(PMQ) \geq v - \frac{1}{2}(v - 1) = \frac{1}{2}(v + 1) \) since at most \( \frac{1}{2}(v - 1) \) of the latter \( d_i \)'s have \( p \) as a factor.

In the case of planes or biplanes of prime order, our next two propositions provide us with an answer in the situation in which the prime of interest is the order of the design.

**Proposition 2.3.** If \( M \) is the incidence matrix of a plane, then \( d_e = (n + 1)n \). Moreover, if \( n = p \), then \( \text{Rk}_p(M) = \frac{1}{2}(v + 1) \).

**Proof.** Now \( d_e = \Delta_e/\Delta_{e-1} \), where
\[
\Delta_i = \gcd \{\det(M_{ij}) | M_{ij} \text{ is an } i \times i \text{ submatrix of } M\}.
\]
A standard calculation, such as the one involved in proving part (4) of Lemma 1.5 (but much longer), yields
\[
\Delta_{e-1} = \gcd \{n^{(1/2)(e-1)}, n^{(1/2)(e-3)}\}.
\]

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Since $\Delta_v = \text{det}(M) = k n^{(1/2)(v-1)} = (n+1) n^{(1/2)(v-1)}$, we obtain $d_v = (n+1) n$. If $n = p$, then, by Proposition 2.2, $\text{Rk}_p(M) \geq v - \frac{1}{2}(v-1) = \frac{1}{2}(v+1)$. Also, the last $\frac{1}{2}(v-1)$'s have $p$ as a factor. Hence $\text{Rk}_p(M) = \frac{1}{2}(v+1)$.

**Proposition 2.4.** If $M$ is the incidence matrix of a biplane, then $d_v = \frac{1}{2}(n+2)n$ if 2 divides $n$ and $(n+2)n$ otherwise. Moreover, if $2 \nmid n = p$, then $\text{Rk}_p(M) = \frac{1}{2}(v+1)$.

**Proof.** Another standard, but tedious, calculation yields

$$\Delta_{v-1} = \gcd\{n^{(1/2)(v-1)}, 2 n^{(1/2)(v-3)}\},$$

from which the results follow as before.

Additional information on the preceding three propositions can be found in [42], [46], and [48].

Thus each of the four biplanes of order 7 must have a 7-rank of 19, and so the concept of $p$-rank cannot aid us in distinguishing these four biplanes. (The 3-rank for each is 36 and the $p$-rank where $p$ is other than 3 or 7 must be 37 by virtue of Theorem 2.1.) We shall see in Section 3 that the notion of $\lambda$-chain is useful in distinguishing them.

Let $q = p^a$ be a prime power. A PG$(t,q)$: $\mu$ 2-design (projective geometry design) is one in which the points are the 1-dimensional subspaces of $F_q^{t+1}$ and the blocks are the $(\mu + 1)$-dimensional subspaces of $F_q^{t+1}$.

PG$(2,q)$:1 is the Desarguesian plane of order $q$, and in this manner we obtain infinitely many planes, one of each prime power order. The plane of Example 1.1 is PG$(2,2)$:1.

For these particular designs, the question of $p$-rank has been completely answered. If $p \nmid n$, then of course consult Theorem 2.1, whereas if $p | n$, see below.

MacWilliams and Mann [38] obtained

$$\text{Rk}_p(M) = \left(\frac{p+1}{2}\right)^a + 1,$$

where $M$ is the incidence matrix for PG$(2,q)$:1, the Desarguesian plane of order $q$. Smith [52], and Goethals and Delsarte [20] obtained

$$\text{Rk}_p(M) = \left(\frac{p+t-1}{t}\right)^a + 1,$$

where $M$ is the incidence matrix for PG$(t,q)$: $\mu$ - 1, the projective design of points and hyperplanes. Hamada (see [29]) obtained the formula for $\text{Rk}_p(M)$, where $M$ is the incidence matrix for PG$(t,q)$: $\mu$, the general projective geometry design. However, it is too complicated to present here.

Hamada [29] has conjectured that $\text{Rk}_p(\text{PG}(t,q):\mu) < \text{Rk}_p(M)$, where $M$ is the incidence matrix for a design with the same parameters as those of PG$(t,q):\mu$, with equality if and only if the design is PG$(t,q)$: $\mu$. Hamada and Ohmori [30] have proved the conjecture in the case of $q = 2$ and $\mu = t - 1$, but their proof does not appear to be generalizable.

Of course, all known examples bear out Hamada’s conjecture. In the case of the four known planes of order 9, the 3-rank of the Desarguesian one is of course 37, and the other three known planes [53], constructed from quasifields, each have a 3-rank of 41 [46]. The author has studied five planes of order 16. The Desarguesian plane has a 2-rank of 82, the three planes constructed from quasifields presented in [36] have a 2-rank of 98 [48], and the fifth plane [37], also constructible from a quasifield, has a 2-rank of 122 [48]. These interesting results for the non-Desarguesian planes of orders 9 and 16 certainly warrant further study.

Since every plane of order $p$ has, by virtue of Proposition 2.3, a $p$-rank of $\frac{1}{2}(v+1) = \left(\frac{p+1}{2}\right) + 1$, Hamada’s conjecture would imply the famous conjecture that the only plane of
order \( p \) is the Desarguesian one, \( \text{PG}(2, p): 1 \). The following theorem examines, in more detail, the code \( A_p \) generated by the incidence matrix of a plane of order \( n \), where we take \( p \) to be a divisor of \( n \), the interesting situation discussed above. The proof of this theorem for Desarguesian planes can be found in [21], and for an arbitrary plane see [46], [47]. Additional information on this subject is also contained in chapter 11 of [16].

**Theorem 2.5.** Let \( M \) be the incidence matrix for a plane of order \( n \) and let \( A_p \) denote, as before, the code generated by the rows of \( M \) over \( F_p \), where \( p \) divides \( n \). Then:

1. The minimum weight of \( A_p \) is \( n + 1 \); that is, \( d(A_p) = k \).
2. \( \text{Wgt}(a) = k \) for \( a \in A_p \) if and only if \( a = \gamma l \) for some nonzero \( \gamma \in F_p \) and \( l \) a row of \( M \).

The support of a row of \( M \) is, of course, a line of the plane. Amazingly, one thus recovers the plane from the \((\mod p)\) span of its incidence matrix simply by taking the supports of the minimum weight vectors. The entire plane can therefore be reconstructed from a basis for the code. This theorem can also be used, of course, in testing whether a collection of \( k \)-subsets can be enlarged to form the collection of lines of a plane.

However, neither part of Theorem 2.5 holds true for biplanes in general, as is illustrated below in the case of the three biplanes of order 4, the projective \((16, 6, 2)\) designs. We first indicate coding theoretic constructions for these three biplanes.

Let \( \text{SP}(\mathbb{B}_i) \) denote the \((16, i)\) code over \( F_2 \) generated by the rows of the incidence matrix \( M \) for the biplane \( \mathbb{B}_i \), where \( i = 6, 7, \) or 8 is the 2-rank of \( M \).

Construct a \( 5 \times 16 \) matrix \( N \) as follows. The first four rows are formed by taking the sixteen columns to be the sixteen vectors in \( F_2^6 \). The last row consists entirely of 1's. The (mod 2) span of \( N \) is the \((16, 5)\) first-order Reed-Muller code, and we denote it by \( H^1 \). The orthogonal of this code, which we denote by \( H \), is the extended binary \((16, 11)\) Hamming code. Since each row of \( N \) contains an even number of 1's and the dot product of any two distinct rows is even, we see that \( H^1 \subseteq H \), and so \( H \) is self-orthogonal.

The weight distribution for \( H^1 \) (where \( n^i \) means \( n \) code vectors of weight \( i \)) is

\[
x^0 + 30x^8 + x^{16}
\]

and the weight distribution for \( H \) is

\[
x^0 + 140x^4 + 448x^6 + 870x^8 + 448x^{10} + 140x^{12} + x^{16}.
\]

**Proposition 2.6.** Let \( F = F_2 \).

1. Let \( v \) be any weight-6 vector of \( H \). Let \( B_6 = H^1 + Fv \). Then the weight distribution for \( B_6 \) is

\[
x^0 + 16x^4 + 30x^8 + 16x^{10} + x^{16}.
\]

Moreover, \( B_6 = \text{SP}(\mathbb{B}_6) \) and the 448 weight-6 vectors of \( H \) split naturally into 28 disjoint \( \mathbb{B}_6 \)'s.

2. Let \( v \) be a weight-6 vector in \( (\text{SP}(\mathbb{B}_6))^1 - \text{SP}(\mathbb{B}_6) \) and let \( B_7 = \text{SP}(\mathbb{B}_6) + Fv \). Then the weight distribution for \( B_7 \) is

\[
x^0 + 4x^4 + 32x^6 + 54x^8 + 32x^{10} + 4x^{12} + x^{16}.
\]

Moreover, \( B_7 = \text{SP}(\mathbb{B}_7) \) and the 32 weight-6 vectors of \( B_7 \) split into two disjoint \( \mathbb{B}_6 \)'s in exactly four ways and two disjoint \( \mathbb{B}_7 \)'s in exactly four ways, thereby yielding the eight \( \mathbb{B}_6 \)'s and the eight \( \mathbb{B}_7 \)'s which can be formed from the 32 weight-6 vectors of \( B_7 \).

3. Let \( v \) be a weight-6 vector in \( (\text{SP}(\mathbb{B}_7))^1 - \text{SP}(\mathbb{B}_7) \) and let \( B_8 = \text{SP}(\mathbb{B}_7) + Fv \). Then the weight distribution for \( B_8 \) is

\[
x^0 + 12x^4 + 64x^6 + 102x^8 + 64x^{10} + 12x^{12} + x^{16}.
\]

Moreover, \( B_8 = \text{SP}(\mathbb{B}_8) \) and the 64 weight-6 vectors of \( B_8 \) split into four disjoint \( \mathbb{B}_6 \)'s, \( \mathbb{B}_7 \)'s, and \( \mathbb{B}_8 \)'s. Also, there exist precisely 96 \( \mathbb{B}_6 \)'s, 288 \( \mathbb{B}_7 \)'s and 192 \( \mathbb{B}_8 \) in \( B_8 \).

We here present a proof of part (1) of this proposition. Proofs of the other parts, along with a
wealth of additional information on these three biplanes, can be found in [7] and [48].

**Proof.** Now \( B_6 \subset B_6^+ \), since \( v \) is in \( H \) and \( H^+ \subset H \). Let \( a \in H^+ \) be a vector of weight-8. Since \( a \cdot v \in \{0, 2, 4, 6\} \), we obtain \( \text{wgt}(a + v) \in \{14, 10, 6, 2\} \). But possibilities 2 and 14 are impossible since \( B_6 \subset H \) which has no vectors of weights 2 or 14. Since the all-one vector is in \( H^+ \), the number of vectors of weights 6 and 10 must be the same. Thus \( B_6 \) has the desired weight distribution.

Let \( b_i \) and \( b_j \) be any two distinct weight-6 vectors in \( B_6 \). Now \( b_i \cdot b_j \in \{0, 2, 4\} \) thereby yielding \( \text{wgt}(b_i + b_j) \in \{12, 8, 4\} \), but \( B_6 \) has no vectors of weights 4 or 12. Hence

\[
[\text{supp}(b_i) \cap \text{supp}(b_j)] = 2,
\]

and so the supports of the 16 weight-6 vectors of \( B_6 \) form the blocks of a biplane (see Proposition 1.8) with a 2-rank of 6; that is, \( B_6 = \text{SP}(\mathcal{B}_6) \). Since each weight-6 vector of \( H \) selects 15 others which together determine the 16 blocks of the biplane, the \( 488 = 16 \cdot 28 \) weight-6 vectors of \( H \) split naturally into 28 disjoint \( \mathcal{B}_6 \)'s.

We know of no other such “nesting” of designs. By Theorem 2.5, such a nesting is clearly impossible for planes of the same order, since the supports of the minimum weight code vectors are precisely the lines of the plane. Of the three biplanes of order 4, only \( \mathcal{B}_6 \) enjoys this property.

3. \( \lambda \)-chains, Difference Sets, and Ovals. In this section we present various notions which are useful in the characterization and construction of projective designs, particularly planes and biplanes. We shall make use of these ideas in Section 4.

We first present Hussain’s manner of describing a biplane in terms of \( \lambda \)-chains.

1. Choose a block, call it the **indexing block**, and index its points from 1 to \( k \).
2. Index each remaining block by the unique 2-subset in which it meets the indexing block.
3. Index each point \( P \) not incident with the indexing block by the collection of \( k \) 2-subsets which index the blocks that contain \( P \). (Each point incident with the indexing block is contained in at least 2 of these 2-subsets.)

4. If the 2-subsets \( \{a_1, a_2\}, \{a_2, a_3\}, \ldots, \{a_{n-1}, a_n\}, \{a_n, a_1\} \) index a collection of blocks incident with \( P \), we express this by writing \( (a_1, a_2, \ldots, a_{n-1}, a_n) \). Each cycle must be at least of length 3. The collection of cycles for \( P \) is called the **\( \lambda \)-chain**, or simply **chain**, for \( P \).

A \( \lambda \)-chain may also be viewed as a graph on the points of the indexing block where the points become the vertices and the 2-subsets become the edges. So each \( \lambda \)-chain becomes a graph of valency 2 which is a disjoint union of polygons.

We call a \( \lambda \)-chain consisting of cycles containing \( c_1, c_2, \ldots, c_m \) letters, respectively, a type \( (c_1 - c_2 - \cdots - c_m) \)-chain. The number of distinct types of \( \lambda \)-chains which are possible for a biplane equals the number of partitions of \( k \) where each summand is at least 3. The **chain structure** of the biplane for a given indexing block is simply the number of chains of each type.

Of course, \( v - k \) chains comprise the chain structure for each choice of indexing block. These chains can be systematically examined by realizing that each pair of 2-subsets \( \{i, j\} \) and \( \{1, 2\} \), where \( 2 < i < j < k \), must be associated with precisely one \( \lambda \)-chain (the blocks indexed by these 2-subsets meet in point “1” and in a point not incident with the indexing block), thereby yielding

\[
v - k = \binom{k - 1}{2}.
\]

Unless the automorphism group of the biplane is transitive on its blocks, the chain structure may depend on the choice of the indexing block. Two biplanes are necessarily nonisomorphic if the two collections of chain structures are not identical.

The \((7, 4, 2)\) biplane of Example 1.2 will have \( v - k = 7 - 4 = 3 \) chains comprising each of its \( b = 7 \) possible chain structures. Each chain is necessarily a 4-cycle. There is only one possible chain associated with \( \{1, 2\} \) and \( \{1, 3\} \), namely \((1, 2, 4, 3)\). (Of course \((1, 2, 4, 3), (2, 4, 3, 1), (3, 4, 2, 1)\), etc., all represent the same chain.) The same is true for the other two pairs of
2-subsets. Thus the only possible distinct chains are \((1, 2, 4, 3), (1, 2, 3, 4), (1, 3, 2, 4),\) and they do determine the biplane of Example 1.2. Each of the seven chain structures must therefore be the same. This argument actually establishes the uniqueness of the biplane of order 2.

For biplanes of order 4, there are two possible chain types, namely, \((6)-\)chains and \((3-3)-\)chains. The number of chains comprising a given chain structure is \(v-k=16-6=10\). Consider the chain associated with the pair of 2-subsets \(\{1, 2\}\) and \(\{1, 3\}\). There are one \((3-3)-\)chain and six \((6)-\)chains from which to choose, and the same is true for each of the other nine pairs of 2-subsets. The reader may verify, quite simply, that the choice of the \((3-3)-\)chain in each instance is a correct choice (each pair of points is contained in precisely two blocks). This yields the biplane whose incidence matrix has a 2-rank of 6, which we have previously called \(\mathcal{B}_6\). Each of the sixteen chain structures for \(\mathcal{B}_6\) consists of these ten \((3-3)-\)chains. For \(\mathcal{B}_7\), each chain structure consists of four \((6)-\)chains and six \((3-3)-\)chains, and for \(\mathcal{B}_8\), each structure consists of six and four, respectively.

As one further example of this \(\lambda\)-chain notion, we will examine one of the biplanes of order 7. The following group-theoretic description can be found in [15]. Let \(\mathcal{P}\) consist of the 1-dimensional subspaces of \(\text{PG}(1, 8)\) (see Section 2). Note that \(\text{PGL}_2(8)\), the projective general linear group of \(2\times 2\) matrices with entries from the field with 8 elements, acts sharply triply transitively on \(\mathcal{P}\) (see [10, Corollary 2.6.5]); that is, given any two ordered triples of elements of \(\mathcal{P}\), there exists a unique element of \(\text{PGL}_2(8)\) which sends the first ordered triple to the second. Let the set of points of the biplane consist of the nine elements of \(\mathcal{P}\) and the \(\binom{9}{3}/3\) subgroups of order 3 of \(\text{PGL}_2(8)\). Each of these 28 subgroups is generated by an element of order 3 with no fixed points, and the orbit structure is that of a chain with all 3-cycles. One block, indexed by \(\mathcal{P}\), consists of the 9 points of \(\mathcal{P}\). Each of the remaining 36 blocks is indexed by a 2-subset of \(\mathcal{P}\) and such a block consists of the 2-subset and the 7 subgroup-of-order-3 points which contain the 2-subset in one of its orbits. Originally discovered by Hussain [34], this biplane’s chain structures, and those of its nonisomorphic dual (presented below on the right), are:

\[
\begin{array}{ccc}
1 & \text{with} & 28 \\
36 & \text{with} & 21 \\
7 & \text{(6-3)-chains} & 1 \\
\end{array}
\]

Although the 7-rank does not aid in distinguishing the four biplanes of order 7, the \(\lambda\)-chain idea does since the collection of chain structures for each biplane is different [49].

Additional information on the chain structures of the known biplanes can be found in [48], [49], and [50].

Let \(D=\{g_1, g_2, \ldots, g_k\}\) be a \(k\)-subset of a group \(G\) with \(v\) elements. If for each nonidentity element \(d\) of \(G\) there exist precisely \(\lambda\) ordered pairs \((g_i, g_j)\) of elements of \(D\) such that \(g_i g_j^{-1} = d\), then we call \(D\) a \((v, k, \lambda)\) group difference set. If \(G=\mathbb{Z}_v\), then \(D\) is simply called a \((v, k, \lambda)\) difference set. (See [13] and [24].)

**Proposition 3.1.** If \(D\) is a \((v, k, \lambda)\) group difference set composed of elements of the group \(G\), then the sets \(Dg=\{gg_1, g_2g, \ldots, g_kg\}\), for all \(g \in G\), form the blocks of a projective \((v, k, \lambda)\) design. (If \(G=\mathbb{Z}_v\), such a design is called a cyclic design.)

**Proof.** Let \(a\) and \(b\) be any two distinct elements of \(G\). We wish to exhibit precisely \(\lambda\) blocks which contain these two elements. Let \(d=ab^{-1}\). There exist precisely \(\lambda\) ordered pairs \((g_i, g_j)\) such that \(g_i g_j^{-1} = d=ab^{-1}\). Thus \(g_i^{-1}a = g_j^{-1}b\) are the \(\lambda\) \(g\)'s such that \(Dg\) contains both \(a\) and \(b\). Since \(v\) blocks are constructed in this manner, we have a projective design.

Every Desarguesian plane is, in fact, a cyclic plane, and no other cyclic planes are known. For the method of constructing these difference sets see [51] and Chapter 11 of [24].

If one relabels “7” as “0” in Example 1.1, one obtains the construction of the plane of order 2.
from the difference set $D = \{1, 2, 4\}$, which consists of the quadratic residues (mod 7) (see [24, p. 141]). Similarly, the biplane of order 2 of Example 1.2 is constructed from the difference set $D = \{1, 2, 3, 6\}$. The unique biplane of order 3 can be constructed via the $(11, 5, 2)$ difference set $D = \{1, 3, 4, 5, 9\}$, which consists of the quadratic residues (mod 11). Also, another one of the biplanes of order 7, discovered independently by Bose [12] and Fisher [19], can be constructed from the difference set which consists of the biquadratic residues (mod 37).

Each of the three biplanes of order 4 can be constructed via $(16, 6, 2)$ group difference sets. (See [35] and [7] for a detailed discussion.) For example, the group difference set

$$D = \{ (0, 0), (0, 1), (0, 2), (0, 5), (1, 0), (1, 6) \}$$

in $\mathbb{Z}_2 \times \mathbb{Z}_8$ [54] yields the biplane of 2-rank 6, which we have called $\mathfrak{B}_6$.

We now turn to the final topic of this section, which is the notion of oval. Although of great interest in the theory of planes [18], the notion of oval has only recently been generalized by Assmus and van Lint [8] to arbitrary projective designs. We present here but a few of their results.

Let $(S, \mathfrak{O})$ be a projective $(v, k, \lambda)$ design. An arc $\mathfrak{A}$ is a subset of $S$ with the property that no three points of $\mathfrak{A}$ lie on a block. That is, for each $B \in \mathfrak{O}$, $|B \cap \mathfrak{A}| = 0$, 1, or 2, and $B$ is called an exterior, tangent, or secant block, respectively.

**Theorem 3.2.** If $(S, \mathfrak{O})$ is a projective $(v, k, \lambda)$ design with $k > 2$ and the design is of either odd order or even order with $k \equiv 0 \pmod{\lambda}$, then $|\mathfrak{A}| < (1/\lambda)(k/\lambda - 1)$. If it is of even order with $k \equiv 0 \pmod{\lambda}$, then $|\mathfrak{A}| < (1/\lambda)(k/\lambda + 1)$.

An arc, $\mathfrak{A}$, of the projective design $(S, \mathfrak{O})$ is called an oval, $\Theta$, whenever it achieves the prescribed bound. We denote the collection of all ovals of the design by Oval $(\mathfrak{O})$.

**Corollary.** Let $n = k - \lambda$ denote, as usual, the order of the projective design.

(1) If $\Theta$ is an oval of a plane of order $n$, then $|\Theta| = n + 1$ if $n$ is odd and $n + 2$ if $n$ is even.

(2) If $\Theta$ is an oval of a biplane of order $n$, then $|\Theta| = \frac{1}{2}(n + 3)$ if $n$ is odd and $\frac{1}{2}(n + 4)$ if $n$ is even.

**Proposition 3.3.** If $\Theta$ is an oval in a projective design of even order with $k \equiv 0 \pmod{\lambda}$, then $\Theta$ has $(1/2\lambda)(k + \lambda)$ secants, $(1/2\lambda)(k - 2)(k - \lambda)$ exterior blocks, and no tangents. If $\Theta$ is an oval in a projective design of odd order, then $\Theta$ necessarily has $(1/\lambda)(k + \lambda - 1)$ tangents, $(1/2\lambda)(k + \lambda - 1)(k - 1)$ secants, and $(1/2\lambda)(k - \lambda - 1)(k - 1)$ exterior blocks. Moreover, in the odd order case, through each point of $\Theta$ there passes exactly one tangent, through each point not on $\Theta$ there pass either two tangents (exterior point) or none (interior point), and thus the tangents form an oval, $\Theta^d$, in the dual design.

Let us now examine the oval structure of some planes and biplanes. If there is a unique biplane of order $n$, we shall denote it by $\mathfrak{B}(n)$.

Let $\Theta$ be an oval of PG$(2, 2)$, the plane of order 2; then $\Theta$ is a 4-subset that does not contain a line and so

$$|\text{Oval(PG}(2, 2))| = \binom{7}{4} - 7 \cdot 4 = 7;$$

that is, Oval (PG$(2, 2)) = \mathfrak{B}(2)$ (see Examples 1.1 and 1.2). Also, an oval $\Theta$ of $\mathfrak{B}(2)$ is a 3-subset not contained in any block of $\mathfrak{B}(2)$, and so

$$|\text{Oval}(\mathfrak{B}(2))| = \binom{7}{3} - 7 \cdot 4 = 7;$$

that is, Oval (\mathfrak{B}(2)) = PG$(2, 2)$.

The oval situation for all known odd order biplanes is as follows.

$n = 1$: $\mathfrak{B}(1)$ is the complete $(4, 3, 2)$ biplane whose blocks are all 3-subsets of a 4-set $S$. Since
an oval contains two elements of $S$, Oval $(\otimes(1))$ consists of the \( \binom{4}{2} = 6 \) 2-subsets of $S$.

$n=3$: Here \(|\emptyset|=3\), and so an oval is a 3-subset which is not contained in a block. Thus

\[
|\text{Oval}(\otimes(3))| = \binom{11}{3} - 11 \cdot \binom{5}{3} = 55.
\]

The Lehigh University CDC 6400 computer was employed in establishing the existence of precisely four biplanes of order 7 ($\lambda$-chains were employed in the exhaustive search), constructing the last biplane of order 7 and one of the four known biplanes of order 9, and examining the oval structure for the biplanes of orders 7, 9, and 11 (see [49] and [50]). (When both the size of an oval and $k$ exceed 3, the situation is of course more complicated than we saw above.) A group-theoretical description of the oldest biplane of order 9 is in [26] and the last two were discovered by R. H. F. Denniston.

$n=7$: The difference set biplane has no ovals. Each of the other three has 63 ovals. In particular, the biplane constructed via \( \text{PGL}_2(8) \) has the block indexed by \( \text{PG}(1, 8) \) tangent to each of its ovals, and this is not true of any other block. This last result is crucial to one of the constructions we present in Section 4.

$n=9$: The 3-ranks of the four known biplanes are 20, 22, 24, and 26, and the number of ovals present are 336, 120, 64, and 48, respectively. The largest number of ovals which have a common tangent is 36. It is interesting to note that, for these four biplanes, as the 3-rank increases, the number of ovals decreases. Also, the chain structures for these biplanes admit cycles of longer length as the 3-rank increases. Perhaps these statements can be shown to be true in general. We shall certainly have occasion to refer to these conjectures again in the conclusion to Section 4. (See [50] for further information on these biplanes and their automorphism groups.)

$n=11$: The two known biplanes, constructed by Aschbacher [2], are duals of each other, and hence, by Proposition 3.3, admit the same number of ovals, which is 77. Here the largest number of ovals which share a common tangent is 55.

4. Planes from Biplanes. We first present four methods of producing self-orthogonal codes via the incidence matrices of projective designs. (See [6].)

**Theorem 4.1.** Let $M$ be the incidence matrix of a projective $(v, k, \lambda)$ design.

1. If \( k \equiv \lambda \equiv 0 \pmod{p} \), then $A$, the rowspace of $M$ over $F_p$, is a self-orthogonal code.
2. If \( p | n \), where $n = k - \lambda$, but \( p \not| k \), let $G$ be the $v \times (v+1)$ matrix whose first column consists of $\sqrt{-k}$'s and whose last $v$ columns are those of $M$, and let $F = F_p$ if $-k$ is a quadratic residue (mod $p$) and $F_2$ otherwise. Then $A$, the rowspace of $G$ over $F$, is a self-orthogonal code. Moreover, if $p^2 \not| n$, then $A = A^\perp$ and so $A$ is a $(v+1, \frac{1}{2}(v+1))$ self-dual code over $F$.
3. If \( k+1 \equiv \lambda \equiv 0 \pmod{p} \), let $G$ be the $v \times 2v$ matrix whose first $v$ columns constitute the identity matrix and whose last $v$ columns are the rowspace of $M$. Then $A$, the rowspace of $G$ over $F_p$, is a $(2v, v)$ self-dual code.
4. If \( p = 2 \), \( \lambda \) is odd, and $k$ is even, let $G$ be the $(v+1) \times (2v+2)$ matrix whose first $v+1$ columns constitute the identity matrix, whose $(v+2)$-column consists of 0 in the first row and 1's elsewhere, and whose last $v$ columns are those of $M$ bordered above by a row of 1's. Then $A$, the rowspace of $G$ over $F_2$, is a $(2v+2, v+1)$ self-dual code.

**Proof.** The proofs of (1), (3), and (4) are straightforward and are left to the reader. (In establishing (4), simply note that $k(k-1) = \lambda(v-1)$ implies that $v$ is odd.)

We present a proof of part (2). $A$ is clearly self-orthogonal. Since $A \subseteq A^\perp$ and \( \dim_F(A^\perp) = v+1 \), \( \text{Rk}_F(G) = \frac{1}{2}(v+1) \). Now \( \text{Rk}_F(G) = \text{Rk}_F(M) \) and, by Proposition 2.2, \( \text{Rk}_F(M) = v - \frac{1}{2}(v-1) = \frac{1}{2}(v+1) \) when \( p^2 \not| n \). Thus \( \text{Rk}_F(G) = \frac{1}{2}(v+1) \) and $A$ is a self-dual code.

Any biplane of even order with $p=2$ is an example for Method (1).
For (2), take any plane and a prime that divides the order. For any plane of order $p$, we again obtain $\text{Rk}_p(M) = \frac{1}{2}(v+1)$ (see Proposition 2.3). It is interesting to note that, should a plane of order 10 exist, this theorem guarantees the 2-rank of its incidence matrix to be $\frac{1}{2}(111+1) = 56$.

By taking $p$ to be 2, any biplane of odd order is an example for Method (3); and for (4), take any plane of odd order.

See [6] for additional examples of this theorem which yield perfect codes. Also in [6], Assmus, Mezzaroba, and Salwach present the following technique for producing planes from biplanes via Method (3).

**Theorem 4.2.** Let $M$ be the incidence matrix of a biplane of odd order with parameters $(v, k, 2)$, and let $I$ be a $v \times v$ identity matrix. Let $G$ be the $v \times 2v$ matrix with the $v$ columns of $I$ preceding those of $M$ and let $G'$ be the $v \times 2v$ matrix with the columns of $M'$, the incidence matrix of the dual biplane, preceding those of $I$. Then over $\mathbb{F}_2$, $\text{SP}(G) = \text{SP}(G')$ and this subspace of $\mathbb{F}_2^{2v}$, call it $A$, is a self-dual code with minimum-weight $d(A) = k + 1$. Moreover, the minimum-weight vectors which are neither the rows of $G$ nor the rows of $G'$ have precisely half of their 1's in the first $v$ coordinates.

**Proof.** $G = (I|M)$ and $G' = (M'|I)$. Since $\lambda = 2$ and $k$ is odd, $M'M = I \pmod{2}$ and thus $G'$ may be obtained from $G$ via the standard row operations involved in computing the inverse of a matrix. Hence $\text{SP}(G) = \text{SP}(G')$. $A$ is self-dual by virtue of Method (3) of Theorem 4.1. Since each row of $G$ or $G'$ has weight $k + 1$, $d(A) < k + 1$. In order to demonstrate the remaining assertions, it suffices to establish that the $\pmod{2}$ sum of $s$ distinct rows of $G$, where $2 < s < \frac{1}{2}(k - 1)$, is a vector with weight greater than $k + 1$, since the same argument also establishes the result for $G'$ and any vector providing a counterexample would necessarily have less than $\frac{1}{2}(k + 1)$'s in either the first or second set of $v$ coordinates.

By induction on $s$, the number of rows, we show the $\pmod{2}$ sum of $s$ rows has weight at least $s(k + 3 - 2s)$. The weight of one row is $1 \cdot (k + 1)$ and the weight of the sum of two rows is $2 \cdot (k - 1)$, thereby meeting the bound. Assuming the result for $s$, the sum of $s + 1$ distinct rows has weight at least

$$s(k + 3 - 2s) + (k + 1) - 4s = (s + 1)(k + 3 - 2(s + 1)),$$

since any row will meet the sum of $s$ rows in at most $2s$ coordinates inasmuch as half of the row is a block of a biplane.

Since $s(k + 3 - 2s)$ is a parabola which is concave downward, when considered as a function of $s$, and for $s = \frac{1}{2}(k + 1)$ we obtain the value of $k + 1$ the same as for $s = 1$, the sum of $s < \frac{1}{2}(k + 1)$ rows produces a vector of weight at least $k + 1$.

Note that for a biplane, $v = \frac{1}{2}(k^2 - k + 2)$, and so

$$2v = k^2 - k + 2 = (k - 1)^2 + (k - 1) + 1 + 1,$$

one more than the number of points in a plane of order $k - 1$. Thus we attempt to extract the even order plane from the $(2v, v)$ code obtained from the odd order biplane by selecting all of the weight-$(k + 1)$ vectors with a 1 at a given coordinate, call this collection $F$, and then determine whether the collection of weight-$k$ vectors obtained by simply ignoring the given coordinate contain all of the lines of a plane of order $k - 1$ on the remaining $k^2 - k + 1$ coordinates. We call this process "contracting on a point." Note: The collection of all weight-$(k + 1)$ vectors need not, in general, form a design.

**Proposition 4.3.** If the collection $F$ yields all of the lines of a plane of order $k - 1$, then these lines "constitute" all of $F$. Moreover, the supports of all other weight-$(k + 1)$ vectors of $A$ are ovals of the plane of order $k - 1$ just obtained.

**Proof.** Let $P$ be the point on which we contract. Let $N$ be the $(k^2 - k + 1) \times (k^2 - k + 2)$ matrix whose rows are the weight-$(k + 1)$ vectors of $F$ which yield the lines of the plane.
A ⊆ (SP(N))⊥. Say there exists \(w \in \mathcal{C}\) such that \(m = \text{supp}(w) - \{P\}\) is not a line. \(|m| = k\). Since \(w \in (\text{SP}(N))^⊥\) and \(w\) meets each row of \(N\) in \(P\), \(m\) must meet each line of the plane in at least one point. Let \(Q, R \in m\). Let \(l\) be the unique line containing \(Q\) and \(R\). Now, there exists \(S \in l - m\) and there exist \(k-1\) lines other than \(l\) through \(S\), each meeting \(m\). Hence \(|m| > (k-1)+2\), a contradiction.

Let \(w \in A\) be such that \(w \notin \mathcal{C}\) and \(\text{wgt}(w) = k + 1\). Let \(Q \subseteq \text{supp}(w)\). Let \(l\) be any of the \(k\) lines through \(Q\). Since \(w \in (\text{SP}(N))^⊥\), \(|l \cap \text{supp}(w)|\) is even, and so each \(l\) meets \(\text{supp}(w)\) in at least one point other than \(Q\). Since \(|\text{supp}(w)| = k + 1\), it must be exactly once more (two lines of a plane meet in precisely one point). Therefore \(\text{supp}(w)\) is an oval of the plane.

Note that the preceding paragraph is the essential ingredient in proving that \(d(\text{SP}(M)^⊥) = n + 2\) and that the weight-(\(n+2\)) vectors of \((\text{SP}(M))^⊥\) are precisely the ovals of the plane of even order \(n\) with incidence matrix \(M\).

Since \(G=(I|M)\) and \(G'=(M'|I)\) both span \(A\) (see Theorem 4.2), we index the first \(v\) columns of \(G\) by the blocks of the biplane and the last \(v\), as usual, by the points. Assmus and van Lint [8] have characterized the minimum-weight vectors of \(A\) which are neither rows of \(G\) nor of \(G'\) in terms of biplane ovals as follows:

**Proposition 4.4.** Let \(\varnothing\) be an oval of a biplane of odd order, then \(\varnothing^d \cup \varnothing\) is the support of a minimum-weight vector of \(A\) (of Theorem 4.2). Conversely, if \(w\) is a minimum-weight vector of \(A\) which has half its \(1\)'s in the first \(v\) coordinates, then \(\text{supp}(w) = \varnothing^d \cup \varnothing\), where \(\varnothing\) is an oval of the biplane.

**Proof.** Let \(\varnothing\) be an oval of the biplane of odd order, then \(\varnothing^d\) consists of its tangents. Form a vector \(w \in \mathbb{F}_2^v\) by placing \(1\)'s in the \(\frac{1}{2}(k+1)\) coordinates indexed by the tangents and the \(\frac{1}{2}(k+1)\) coordinates indexed by the points of \(\varnothing\). We now show that \(w \in A\). Since through each point of \(\varnothing\) there passes exactly one tangent and through each point not on \(\varnothing\) there pass two tangents or none (see Proposition 3.3), the (mod 2) sum of the rows of \(G\) indexed by the tangents will yield a vector whose support is \(\varnothing^d \cup \varnothing\). Therefore \(\varnothing^d \cup \varnothing\) gives rise to a minimum-weight vector of \(A\) which is of the desired form.

Conversely, let \(w\) be a minimum-weight vector of \(A\) of the correct form. Let \(\varnothing\) be the support of the last \(v\) coordinates of \(w\). We now show that \(\varnothing\) is a biplane oval. Let \(P \in \varnothing\). There exist \(\frac{1}{2}(k-1)\) other points in \(\varnothing\). Since through \(P\) and each other point of \(\varnothing\) there pass two blocks of the biplane, at most \(k-1\) blocks through \(P\) are accounted for. Hence there exists at least one block \(B\) such that \(B \cap \varnothing = \{P\}\). Since \(A=A^⊥\), there is a 1 in the coordinate of \(w\) indexed by \(B\). Thus each of the \(\frac{1}{2}(k+1)\) 1's in the first \(v\) coordinates of \(w\) are accounted for by a single such \(B\) for each \(P \in \varnothing\). Hence each such \(B\) is unique, and so each of the remaining \(k-1\) blocks through \(P\) must meet \(\varnothing\) in precisely one more point. Thus \(\varnothing\) is an oval of the biplane and the support of the first \(v\) coordinates of \(w\) forms \(\varnothing^d\).

We conclude by examining the odd order biplanes in light of this construction. See [6], [8], [42], [48], and [50] for more detailed information on these intriguing examples.

\(n=1\): The (4, 3, 2) biplane yields, via Theorem 4.2, the doubly even self-dual extended binary (8, 4) Hamming code whose 14 weight-4 vectors form the 3-(8, 4, 1) design of Example 1.3 and we show this as follows: Two distinct weight-4 vectors meet in either zero or two 1's and so each 3-subset of the \(\binom{8}{3}=56\) possible is contained in the support of at most one weight-4. But each of the 14 weight-4's has \(\binom{4}{3}=4\) 3-subsets yielding a total of 56, so each 3-subset is contained in the support of precisely one weight-4. This 3-(8, 4, 1) design is the unique extension of the plane of order 2, the 2-(7, 3, 1) design of Example 1.1. Contracting on any point yields the plane of order 2. Also, as we saw in Section 3, \(\varnothing(1)\) has 6 ovals. These 6 (by virtue of Proposition 4.4) together with the 4 rows of \(G\) and the 4 rows of \(G'\) yield the 14 weight-4 vectors.
The (11,5,2) biplane yields a (22,11) self-dual code \( A \) with \( d(A)=6 \). We show that the collection of all weight-6 vectors forms a 3-(22,6,1) design, and so will be the extension of the plane of order 4, a 2-(21,5,1) design. Let \( b_i \) and \( b_j \) be any two distinct vectors of weight 6 in \( A \).

\[
|\text{supp}(b_i) \cap \text{supp}(b_j)| < 2
\]

since if it were at least 3, then, because \( A=A^\perp \), it would be 4, but then \( \text{wgt}(b_i+b_j)=4<6=d(A) \), a contradiction. Thus each 3-subset is in the support of at most one weight-6 vector. Since the rows of \( G \) and \( G' \) yield 22 weight-6 vectors, and \( \mathcal{B}(3) \) admits 55 ovals, thereby yielding, via Proposition 4.4, 55 weight-6 vectors which “split 3-3”, and \( 77 \cdot \left( \begin{array}{c} 6 \\ 3 \end{array} \right) = \left( \begin{array}{c} 22 \\ 3 \end{array} \right) \), we see that \( A \) contains 77 weight-6 vectors and that each 3-subset is in the support of precisely one weight-6 vector.

For \( n=7 \), all four (37,9,2) biplanes yield (74,37) self-dual codes. The difference set biplane cannot yield the plane of order 8, since difference set projective designs have transitive automorphism groups and are isomorphic to their duals. Hence the (74,37) code of this biplane has a transitive automorphism group and so, if a plane was obtained by contracting at one point, it would be obtained by contracting at any point, a contradiction, since only the planes of orders 2 and 4 (and possibly 10, if it exists), have extensions (see Theorem 1.11). Only the biplane gotten via \( \text{PGL}_2(8) \) yields the plane of order 8, and only by contracting on the first coordinate (indexed by \( \mathcal{P} = \text{PG}(1,8) \)). Since \( 73=1+9+63 \), we yet need 63 weight-10 vectors which split 5-5. Since \( \text{PGL}_2(8) \) acts sharply triply transitively on \( \mathcal{P} \), there exist \( 9 \cdot \left( \begin{array}{c} 8 \\ 2 \end{array} \right) / 4 = 63 \) subgroups of order 2, each of whose orbits consist of a fixed point and four 2-subsets. The (mod 2) sum of the rows indexed by \( \mathcal{P} \) and the four 2-subsets yield a 5-5 weight-10 vector with a 1 in the first coordinate. These 63 weight-10 vectors can, of course, be constructed as well from the 63 ovals of the biplane, each of which has as a tangent the block indexed by \( \mathcal{P} \). These 73 vectors do yield the plane of order 8 on coordinates 2 through 74 (see [48]). It is interesting to note that we contracted on the unique indexing block that yielded all (3-3-3)-chains. Of course, the chain structure for \( \mathcal{B}(1) \) consists of a (3)-chain, and for \( \mathcal{B}(3) \) it consists of (5)-chains.

One of the four known (56,11,2) biplane (in fact, the one presented in [26]) can be obtained via the 56 = 77 - 21 unused weight-6 vectors in the case of \( n=3 \) when contracting to produce the plane of order 4; that is, the 56 weight-6 vectors with a 0 at a given coordinate. Let \( b_i \) and \( b_j \) be any two such vectors. Define \( M \) by \( m_{ij} = 0 \) if \( b_i \cdot b_j = 2 \) and 1 otherwise. \( M \) is the incidence matrix of the biplane of order 9 whose 3-rank is 20. Unfortunately, none of the four known biplanes of order 9 yield the plane of order 10, since none admit enough ovals. In fact, as we previously observed in Section 3, the number of ovals of a biplane of order 9 seems to decrease as the 3-rank increases and as the length of the \( \lambda \)-chain cycles increase. The same is also true for the biplanes of order 4 [7], [8]. An exhaustive computer search demonstrated the biplanes of 3-rank 20 and 22 to be the only biplanes of order 9 with at least one chain structure consisting entirely of (4-4-3)-chains (see [50]). Thus hopes of producing a plane of order 10 via these techniques do not seem very bright.

The two known (79,13,2) biplanes are duals of each other and do not yield the plane of order 12. Again, the biplanes do not admit enough ovals. A computer search has shown that there does not exist a biplane of order 11 with at least one chain structure consisting entirely of (4-3-3-3)-chains, the seemingly most interesting case.

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SOME HISTORICAL REMARKS CONCERNING DEGREE THEORY

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Summary. It is well known that the notion of topological degree, \( \deg(F) \), of a continuous mapping \( F: M \to N \), where \( M \) and \( N \) are connected, oriented, compact \( n \)-manifolds (with triangulation), can be traced back to L. E. J. Brouwer [Brouwer, 1911, 1912]. A related notion is that of the winding number, \( w(F,0) \), of a continuous mapping \( F: M \to \mathbb{R}^{n+1} - \{0\} \), where \( w(F,0) = \deg(F/|F|) \). In the differentiable case this concept was known as the “Kronecker characteristic” or the “Kronecker integral” of the mapping \( F \) even before Brouwer’s work. The aim of this note is to examine the historical roots of the Kronecker characteristic and some of its applications.

1. Introduction. A familiar representation of the Kronecker characteristic is given by the following integral; see, e.g., [Hadamard, 1910] and [Alexandroff and Hopf, 1935]:

Let \( F: M \to \mathbb{R}^{n+1} - \{0\} \) be a smooth map, and let \( M \) be a smooth, compact, and oriented manifold.

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