THE MOD-2 COHOMOLOGY OF THE BIANCHI GROUPS

ETHAN BERKOVE

ABSTRACT. The Bianchi groups are a family of discrete subgroups of $PSL_2(\mathbb{C})$ which have group theoretic descriptions as amalgamated products and HNN extensions. Using Bass-Serre theory, we show how the cohomology of these two constructions relates to the cohomology of their pieces. We then apply these results to calculate the mod-2 cohomology ring for various Bianchi groups.

1. INTRODUCTION

The Bianchi groups are the groups $\Gamma_d = PSL_2(\mathcal{O}_d)$, where $\mathcal{O}_d$ is the ring of algebraic integers of $\mathbb{Q}(\sqrt{-d})$, and $d$ is any positive square-free integer. They can be thought of as generalizations of $PSL_2(\mathbb{Z})$, the modular group. These groups are classical objects, investigated as early as 1892 by Luigi Bianchi, who built fundamental domains for various values of $d$. Besides their obvious group theoretic interest, the Bianchi groups have been studied from other points of view, some as diverse as 3-manifold theory, the theory of automorphic forms [8], and topological $K$-theory [4].

In this paper we focus on the calculation of the Bianchi groups’ cohomology rings. The difficulty in these calculations lies in that the Bianchi groups are infinite, for $\mathbb{Z} \subset \mathcal{O}_d$ and hence $PSL_2(\mathbb{Z}) \subset PSL_2(\mathcal{O}_d)$. Many of the standard computational techniques of cohomology of groups work best for finite groups, as they depend on knowledge of Sylow $p$-subgroups or restriction maps to detecting subgroups. Such structures in infinite groups may be too complicated to work with or too difficult to find. The key fact that makes calculation possible for the Bianchi groups is that although these groups are infinite, they are built in stages out of relatively simple subgroups welded together through amalgamations and HNN extensions. We show how amalgamations and HNN extensions affect cohomology in general, then build the cohomology rings of the Bianchi groups in stages.

Some work in this vein has been done separately by Alperin [2], Mendoza [9], and Schwermer and Vogtmann [10]. Alperin calculated the integral homology of $SL_2(\mathbb{O}_3)$ using a simplicial complex based on the group itself. Mendoza constructed a two-dimensional deformation retract of hyperbolic three-space which he used to perform various cohomology calculations with module coefficients. Schwermer and Vogtmann used this Mendoza complex to calculate the integral homology of the five Euclidean Bianchi groups, so named because the rings $\mathcal{O}_d$ for these groups have a Euclidean algorithm. Vogtmann also used the Mendoza complex to find the rational cohomology of all Bianchi groups with $d < 100$ [14].

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Our approach uses group presentations, which provides both the mod-2 cohomology ring and its structure over the Steenrod algebra. This approach better shows how the cohomology classes fit together, as well as the extent to which the cohomology of finite subgroups controls the cohomology of the whole group. It also affords a check on the additive structure determined by Schwermer and Vogtmann in [10].

We do all calculations with coefficients in the field of two elements, $\mathbb{F}_2$, unless otherwise indicated. This is an acceptable reduction for a couple of reasons. First, the calculations simplify considerably. Second, the only possible torsion elements in any Bianchi group have order two or three. From the list of finite subgroups one sees that most of the torsion is in fact of order two. Thus, the choice of $\mathbb{F}_2$ coefficients still yields the majority of the total cohomological information about the Bianchi groups.

This paper is organized as follows: In the next section we develop the machinery we will need for the calculations. In the following three sections we do the actual cohomology calculations to get complete ring structures for the five Euclidean cases and three non-Euclidean ones. We cover the groups $\Gamma_6$ and $\Gamma_2$ in detail, then give rough outlines for the remaining cases.

The results in this paper are part of my doctoral dissertation completed at the University of Wisconsin, Madison under Alejandro Adem. He initially pointed me towards the Bianchi groups, and I thank him for his many helpful comments and suggestions.

2. Structure Results

The Bianchi groups are less exotic than they might at first appear. As $\mathcal{O}_d$ is a ring of integers, the Bianchi groups are generalizations of $PSL_2(\mathbb{Z})$. For any square-free $d$, the ring $\mathcal{O}_d$ has a particularly nice algebraic form: as a $\mathbb{Z}$-module, $\mathcal{O}_d \cong \mathbb{Z} \oplus \mathbb{Z}\omega$, where $\omega = \sqrt{-d}$ for $d \equiv 1, 2 \mod 4$ and $\omega = \frac{1+\sqrt{-d}}{2}$ for $d \equiv 3 \mod 4$. Elements of a Bianchi group have a geometric interpretation as well. An element of $\Gamma_d$, $g = \begin{pmatrix} A & B \\ d & C \end{pmatrix}$ with entries in $\mathcal{O}_d$, can be identified with the fractional linear transformation $\frac{Az+B}{Cz+D}$ [3]. In this way $g$ acts by isometries on the Riemann sphere, $\mathbb{C} \cup \infty$, and this action can be uniquely extended to the hyperbolic upper half space $\mathbb{H}^3 = \{(z,\zeta) \in \mathbb{C} \times \mathbb{R}^+ \mid \zeta > 0\}$. The Bianchi groups act properly discontinuously on $\mathbb{H}^3$, so one might hope to use the quotient space for calculations. Unfortunately, the quotient is open and hence not compact, making homology and cohomology calculations difficult. Mendoza overcame this problem by constructing an equivariant deformation retract of $\mathbb{H}^3$ with compact quotient and CW structure [9]. With this complex and the stabilizers of the vertices and edges, one can calculate cohomology rings.

We take a more group theoretic approach, using presentations. The Bianchi groups in this article are built up from infinite cyclic and finite subgroups. This latter family is surprisingly small: one can show that the only possible finite subgroups of a Bianchi group are the cyclic groups of order 2 and 3, denoted $\mathbb{Z}/2$ and $\mathbb{Z}/3$; the elementary abelian group $D_2 \cong \mathbb{Z}/2 \times \mathbb{Z}/2$; the symmetric group on three letters $S_3$; and the alternating group on four letters $A_4$ [8]. Subsets of this family appear in each of the Bianchi groups considered in this paper, and $\Gamma_1$ actually contains them all. Since the cohomology of the Bianchi groups depends on these finite
subgroups, it is appropriate that we start with a description of these well-known mod-2 cohomology rings.

Theorem 2.1 ([1]). The cohomology rings for finite subgroups of Bianchi groups are:

\[ H^*(\mathbb{Z}/3) \cong H^*(1) \cong \mathbb{F}_2, \]
\[ H^*(\mathbb{D}_2) \cong \mathbb{F}_2[x_1, y_1], \]
\[ H^*(\mathbb{Z}/2) \cong H^*(\mathbb{S}_3) \cong \mathbb{F}_2[x_1], \]
\[ H^*(\mathbb{A}_4) \cong \mathbb{F}_2[u_2, v_3, w_3]/(u_2^3 + v_3^2 + w_3^2 + v_3w_3). \]

(The subscripts denote the degree of the generators.)

We need the restriction maps in cohomology between these groups and their subgroups. Most are straightforward, but one case, \( \text{res}_{\mathbb{Z}/2}^{\mathbb{A}_4} \), is a little more involved.

Lemma 2.2. The map \( \text{res}_{\mathbb{Z}/2}^{\mathbb{A}_4} \) is given on a suitable choice of generators by:

\[ \text{res}_{\mathbb{Z}/2}^{\mathbb{A}_4} u_2 = x_1^2, \quad \text{res}_{\mathbb{Z}/2}^{\mathbb{A}_4} w_3 = x_1^3, \quad \text{res}_{\mathbb{Z}/2}^{\mathbb{A}_4} v_3 = 0. \]

Proof. \( \mathbb{D}_2 \), the Sylow two subgroup of \( \mathbb{A}_4 \), fits into the short exact sequence

\[ 1 \rightarrow \mathbb{D}_2 \rightarrow \mathbb{A}_4 \rightarrow \mathbb{Z}/3 \rightarrow 1. \]

As \( \mathbb{D}_2 \) is normal in \( \mathbb{A}_4 \) there is an action of \( \mathbb{Z}/3 \) on \( \mathbb{D}_2 \) with \( H^*(\mathbb{A}_4) = H^*(\mathbb{D}_2)^{\mathbb{Z}/3} \), the invariants under this action. If \( x \) and \( y \) are the generators of \( H^1(\mathbb{D}_2) \), \( H^*(\mathbb{A}_4) \) is generated by the following three classes [1]:

\[ u_2 = x^2 + xy + y^2, \]
\[ v_3 = x^2y + xy^2, \]
\[ w_4 = x^3 + x^2y + y^3. \]

Now choose a copy of \( \mathbb{Z}/2 \) in \( \mathbb{A}_4 \). Since all elements of order two are conjugate, we can assume without loss of generality that \( x \) is the generator of \( H^1(\mathbb{Z}/2) \). Then \( \text{res}_{\mathbb{Z}/2}^{\mathbb{A}_4} x = x, \text{res}_{\mathbb{Z}/2}^{\mathbb{A}_4} y = 0 \), and the result follows. One can easily calculate the action of the Steenrod squaring operations on the generators of \( H^*(\mathbb{A}_4) \) from this description. \( \square \)

The Bianchi groups are formed in stages from their finite subgroups using two group theoretic constructions, the HNN extension and the amalgamated product. The latter is well known to topologists as well as algebraists, but the HNN extension is a little more unusual. We will use it often in our calculations, and so we review its definition, following the notation in Fine’s book [6].

Definition. Let \( G_1 \) be a group, \( G_2 \) a subgroup, and \( \theta : G_2 \rightarrow G_1 \) a monomorphism. Then an HNN extension of \( G_1 \) is a group

\[ G = \langle t, G_1 \mid t^{-1}gt = \theta(g), g \in G_2 \rangle. \]

\( G \) is denoted by \( \text{HN}(t, G_1, G_2, G_2) \). \( G_1 \) is called the base and \( G_2 \) is called the associated subgroup.

Calculating the cohomology of an HNN extension is not much harder than calculating the cohomology of an amalgamated product if one uses the Bass-Serre theory of trees [11]. Summarizing, a tree is a contractible CW complex consisting of zero- and one-cells. Bass and Serre show, for every amalgamated product, how
to construct a tree and action which has a line segment as a quotient, with the factor groups as the isotropy of the vertices and the amalgamated subgroup as the isotropy of the edge. Similarly, for HNN extensions the analogous construction yields an edge and a single vertex, i.e. a cycle, as the quotient. In this case, the associated subgroup is the isotropy of the edge and the base is the isotropy of the vertex (see Figure 1).

After some homological algebra one can derive the following long exact sequence in cohomology.

**Theorem 2.3** ([11]). For $G$ as above, there is a long exact sequence in cohomology

$$
\cdots \rightarrow \bigoplus_{e \in \Sigma_1} H^{i-1}(G_e) \xrightarrow{\delta} H^i(G) \xrightarrow{\beta} \bigoplus_{v \in \Sigma_0} H^i(G_v) \xrightarrow{\alpha} \bigoplus_{e \in \Sigma_1} H^i(G_e) \xrightarrow{\delta} \cdots
$$

The direct sum is over one edge and two vertices if $G$ is an amalgamated product, and over one edge and one vertex if $G$ is an HNN extension.

**Proposition 2.4** ([11]). In the long exact sequence of Theorem 2.3, the map $\beta$ is the restriction map. When $G = G_1 \ast_H G_2$, $\alpha$ is the difference of restriction maps, $\alpha = res^*_{G_1} - res^*_{G_2}$. When $G = HNN(t, G_1, G_2, G_2)$, the “twisting” induced by $\theta$ comes into play, and $\alpha = res^*_G - \theta^*$.

This is enough theory to calculate the cohomology of an amalgamated product or HNN extension as a graded group. In fact, we can recover a good deal of the ring structure as well.

**Proposition 2.5.** For $G$ either an HNN extension or an amalgamated product, the kernel of $\alpha$ in Theorem 2.3 is closed under cup products. That is, for $u, v \in H^*(G_v)$, if $\alpha(u) = \alpha(v) = 0$ then $\alpha(u \cup v) = 0$.

**Proof.** We do the proof for the case of an HNN extension. If $\alpha(u) = 0$, then $res^*(u) = \theta^*(u)$, and likewise for $v$. By naturality of the restriction map and the conjugation map,

$$
res^*(u \cup v) = res^*(u) \cup res^*(v)
$$

$$
= \theta^*(u) \cup \theta^*(v)
$$

$$
= \theta^*(u \cup v).
$$

\[\square\]
Remark 2.6. The map $\beta$ in Theorem 2.3 is a restriction map, so it respects cup products. Thus, as $H^*(G) \xrightarrow{\beta} \bigoplus H^*(G_v) \xrightarrow{\alpha} \bigoplus H^*(G_e)$ is exact, $\text{im}(\beta) \cong \ker(\alpha)$, and this is an isomorphism of rings.

In the calculations to follow, most classes in $H^*(G)$ have non-trivial images under $\beta$, so most of the cohomology ring structure in the Euclidean Bianchi groups comes directly from the cohomology ring structure of the $G_v$, the finite subgroups. At this point we add some notation. As restriction maps are induced by an inclusion map, $i$, we use $i^*$ to mean the restriction map on cohomology. We also need one more result, to determine products with classes which arise from the image of $\delta$.

**Proposition 2.7** (See 5.6 in [12]). In Theorem 2.3, the map $\delta$ respects the cup products in $G$. Let $i : G_v \to G$ be the inclusion map. Then for $u \in H^p(G)$ and $v \in H^q(G_e)$ we have $\delta(i^* u \cup v) = u \cup \delta(v)$ and $\delta(v \cup i^* u) = \delta(v) \cup u$.

These sequences and maps can be developed from other points of view. Using the trees associated to amalgamated products and HNN extensions, the equivariant spectral sequence associated to the quotient space collapses at the $E_2$ page to yield these results. For the amalgamated product, $G = G_1 *_{H} G_2$, it is also possible to use the commutative diagram of $K(\pi, 1)$'s:

$$
\begin{array}{ccc}
H & \longrightarrow & G_1 \\
\downarrow & & \downarrow \\
G_2 & \longrightarrow & G
\end{array}
$$

where all the maps are injections [5]. This construction yields a Mayer-Vietoris sequence, and the properties follow. This point of view can facilitate some results, for example, an analog to the cohomology of a pointed sum of spaces:

**Theorem 2.8** ([5]). The cohomology ring of a free product $G_1 \ast G_2$ with any coefficients $R$ is the reduced sum of the cohomology rings of $G_1$ and $G_2$, namely $H^*(G_1; R) \tilde{\oplus} H^*(G_2; R)$.

**Corollary 2.9.** As $PSL_2(\mathbb{Z}) \cong \mathbb{Z}/2 \ast \mathbb{Z}/3$, $H^*(PSL_2(\mathbb{Z})) \cong H^*(\mathbb{Z}/2)$.

3. THE BIANCHI GROUPS $\Gamma_2$ AND $\Gamma_6$

In this section we use results from the previous section to calculate $H^*(\Gamma_2)$ and $H^*(\Gamma_6)$. These are the most interesting (cohomologically) of the cases we consider so we do them in detail; with the exception of $\Gamma_3$, the other Bianchi groups pose no new challenges.

3.1. The Case $\Gamma_2$. Using Flöge’s presentation for the group, $\Gamma_2 = \langle A, V, S, M, U; A^2 = S^3 = (AM)^2 = M^2 = V^3 = 1, AM = SV^2, U^{-1}AU = M, U^{-1}SU = V \rangle$ [7]. Set $G = \langle A, V, S, M; A^2 = M^2 = (AM)^2 = S^3 = V^3 = 1, AM = SV^2 \rangle$ and consider the subgroups

$$
\begin{align*}
G_1 &= \langle A, M; A^2 = M^2 = (AM)^2 = 1 \rangle \cong D_2, \\
G_2 &= \langle S, V; S^3 = V^3 = (SV^2)^2 = 1 \rangle \cong A_4, \\
H &= \langle AM = SV^2, (AM)^2 = 1 \rangle \cong \mathbb{Z}/2.
\end{align*}
$$

Then $G$ is the amalgamated product $G_1 *_{H} G_2 \cong D_2 *_{\mathbb{Z}/2} A_4$ and $\Gamma_2 = HNN(U, G, PSL_2(\mathbb{Z}), PSL_2(\mathbb{Z}))$, where $PSL_2(\mathbb{Z}) = \langle A, S \rangle$. 


By Theorem 2.3 the cohomology of $G$, an amalgamated product, can be determined from the long exact sequence

$$0 \to H^0(G) \xrightarrow{\beta} H^0(D_2) \oplus H^0(A_4) \xrightarrow{\alpha} H^0(Z/2) \xrightarrow{\delta} H^1(G) \xrightarrow{\beta} \cdots.$$  

The subgroup $H \cong Z/2$ is generated by $AM$, an element of order two which injects into both $A_4$ and $D_2$. Let $z_1 \in H^1(Z/2)$ denote the dual of $AM$ and let $x_1$ and $y_1 \in H^1(D_2)$ denote the duals of the group elements $A$ and $AM$. Say that $H^*(A_4)$ is generated by $u_2$, $v_3$, and $w_3$. By Lemma 2.2, the map $\alpha$ is defined on these generators by:

$$\alpha(x_1) = \alpha(v_3) = 0, \quad \alpha(y_1^k) = z_1^k, \quad \alpha(u_2^k) = z_1^{2k}, \quad \alpha(w_3^k) = z_1^{3k}.$$  

Also, $\alpha(x_1^j y_1^k) = 0$ for all $j > 0$ by the naturality of the cup product. As $\alpha$ is a surjection in all degrees, the long exact sequence (3.1) breaks into short exact sequences

$$0 \to H^k(G) \xrightarrow{\beta} H^k(D_2) \oplus H^k(A_4) \xrightarrow{\alpha} H^k(Z/2) \to 0.$$  

This implies $H^*(G) \cong \text{ker}(\alpha)$, and by Remark 2.6 this is an isomorphism of rings. We identify five classes which we claim generate ker($\alpha$) as a ring: $\bar{x}_1 = (x_1, 0)$, $\bar{y}_2 = (y_2, y_2)$, $\bar{w}_3 = (y_3, w_3)$, and $\bar{v}_3 = (0, v_3)$. The first four are the minimum required to build the classes $(x_1^j y_1^k, 0)$ in the kernel, and $\bar{v}_3$ is clearly necessary as well. Relations among the five are easy to find, $\bar{x}_1^2 \bar{v}_2 = \bar{y}_2^2$, for example, although finding a minimal set takes a little work. This is facilitated by using the Poincaré series, $P(G, t) = \sum_{n \geq 0} \text{dim}_{F_2} H^n(G)t^n$. It is easily determined by $\alpha$, and can subsequently be used as a check to confirm that all generators and relations have been found. In this case,

$$P(G, t) = \frac{1}{(1-t)^2} + \frac{1 + t^3}{(1-t^2)(1-t^3)} - \frac{1}{1-t} = \frac{t^4 + 3t^3 + 2t^2 + t + 1}{(1-t^3)(1-t^2)}.$$  

With some effort we find the relations that agree with the series, which we state in the following lemma. We remove the bars on the classes.

**Lemma 3.1.** $H^*(G) \cong F_2[x_1, y_2, u_2, v_3, w_3]/R$, where $R$ is generated by the set of relations $u_2^3 + v_3^3 + w_3^3 + 3w_3 = 0$, $x_1^2 u_2 = y_2^2$, $x_1 w_3 = u_2 y_2$, $y_2 w_3 = x_1 u_2^2$, and $x_1 v_3 = y_2 v_3 = 0$.

Most of the Steenrod squaring operations are clear:

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<tr>
<th>$Sq^1$</th>
<th>$x_1$</th>
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To calculate $H^*(\Gamma_2)$ from $H^*(G)$ we use the long exact sequence for an HNN extension from Theorem 2.3:

$$0 \to H^0(\Gamma_2) \xrightarrow{\beta} H^0(G) \xrightarrow{\alpha} H^0(PSL_2(Z)) \xrightarrow{\delta} H^1(\Gamma_2) \xrightarrow{\beta} H^1(G) \xrightarrow{\alpha} H^1(PSL_2(Z)) \xrightarrow{\delta} \cdots.$$  

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By Proposition 2.4, \( \alpha \) is the difference of a restriction map \( i^* \) and a twisting map \( \theta^* \). Let \( w_1 \in H^1(PSL_2(\mathbb{Z})) \) denote the generator dual to \( A \in PSL_2(\mathbb{Z}) \). The element \( A \) is also an element of \( D_2 \), which injects into \( G \), so \( i^* \) sends \( x_1 \in H^1(G) \) to \( w_1 \), and all other generators to zero. The twisting map, \( \theta^* \), is more interesting. Three torsion is not detected by \( F_2 \) coefficients, so we only need consider the twisting component in \( D_2 \). Recall that \( U^{-1}AU = M = A \cdot AM \). Therefore \( \theta^* \) sends both \( x_1 \) and \( y_1 \) to \( w_1 \), and hence sends \( x_1^j y_1^k \) to \( w_1^{j+k} \). Both \( \theta^* \) and \( i^* \) are zero on \( H^*(A_4) \) so \( \alpha \) is too.

On \( H^*(D_2) \) we have \( \alpha(x_1^j) = 0 \), and \( \alpha(x_1^j y_1^k) = w_1^{j+k} \) for \( k > 0 \). Summarizing, \[
\begin{align*}
\alpha(x_1) &= \alpha(x_1,0) = 0, \quad \alpha(v_3) = \alpha(0,v_3) = 0, \\
\alpha(y_2) &= \alpha(x_1 y_1,0) = w_1^2, \quad \alpha(u_2) = \alpha(y_1^2, u_2) = w_1^2 + 0 = w_1^2, \\
\alpha(w_3) &= \alpha(y_1^3,w_3) = w_1^3 + 0 = w_1^3.
\end{align*}
\]

Note here that \( \ker(\alpha) \) is not a module over \( H^*(G) \). That is, even though \( \alpha(x_1) = 0 \), we have that \( \alpha(x_1 y_1,0) = w_1^2 \), not 0, as one might expect. One can confirm that the classes \( n_1 = x_1 = (x_1,0), m_2 = y_1 + u_2 = (x_1 y_1 + y_1^2, u_2), m_3 = x_1 u_2 + w_3 = (x_1 y_1^2 + y_1^3, w_3) \), and \( n_3 = v_3 = (0,v_3) \) all lie in the kernel of \( \alpha \).

A Poincaré series argument shows that these classes generate most of the kernel as a ring, subject to the relations \( n_1 n_3 = m_3^2 + m_2 n_3^2 + m_3 n_1 m_3 n_3 = 0 \).

In degrees two and higher \( \alpha \) is a surjection, so \( H^*(\Gamma_2) \) is entirely detected on the kernel of \( \alpha \) from this point on, as the long exact sequence breaks into short exact sequences

\[ 0 \rightarrow H^k(\Gamma_2) \xrightarrow{\beta} H^k(G) \xrightarrow{\alpha} H^k(PSL_2(\mathbb{Z})) \rightarrow 0. \]

There are a couple of classes in \( H^*(\Gamma_2) \) to account for that arise when \( \alpha \) is not surjective. In degrees zero and one \( \alpha \) factors through zero, yielding \( \delta(1) \in H^1(\Gamma_2) \), which we call \( \sigma_1 \) and \( \delta(w_1) \in H^2(\Gamma_2) \) which we call \( \sigma_2 \). The class \( \delta(1) \) will appear in all HNN extensions, as it arises from the geometry of the group (the map \( \beta \) is always an isomorphism at the zero level). These \( \text{HNN classes} \) are exterior, for \( \alpha \) always factors through zero in degree 0 in the long exact sequence for an HNN extension, even with integral coefficients. In this latter case, the resulting class is exterior for dimensional reasons. The class \( \delta(1) \) corresponds to this class under the universal coefficient theorem, and so will be exterior as well.

To determine other products, notice that \( \delta \) factors through zero in degrees two and higher, and that all products with the \( \sigma_1 \) must lie in the image of \( \delta \) by the naturality of \( \beta \). Thus \( \sigma_2^2 = \sigma_1 \sigma_2 = 0 \), leaving \( \sigma_1 n_1 = \sigma_2 \) as the only possible product. That this product is non-trivial follows from Proposition 2.7. The class \( n_1 = (x_1,0) \in H^1(\Gamma_2) \) is dual to the group element \( A \). \( A \) has dual cohomology classes \( x_1 \in H^1(D_2) \) and \( w_1 \in H^1(PSL_2(\mathbb{Z})) \). Thus \( i^*(n_1) = w_1 \), where \( i : PSL_2(\mathbb{Z}) \rightarrow \Gamma_2 \) is the injection map. Then \( \sigma_2 = \delta(w_1) = \delta(1 \cup w_1) = \delta(1 \cup i^*(n_1)) = \delta(1) \cup n_1 = \sigma_1 \cup n_1 \).

A calculation of the Poincaré series confirms that we have the complete ring. We keep the classes \( \sigma_1 \) and \( \sigma_2 \) separate from the calculations for readability. We get

\[
P(\Gamma_2,t) = \frac{t^4 + 3t^3 + 2t^2 + t + 1}{(1-t^2)(1-t^3)} - \frac{t^2}{(1-t)} + t + t^2 \]
\[
= \frac{t^6 + t^5 + t^4 + 3t^3 + t^2 + t + 1}{(1-t^2)(1-t^3)} + t + t^2.
\]
Theorem 3.2. \( H^*(\Gamma_2) \cong F_2[n_1, m_2, n_3, m_3](\sigma_1, \sigma_2)/R \), where \( R \) is generated by the set of relations \( m_2^2 + m_2^2 + n_3^2 + m_3n_3 + n_1m_2m_3 = 0 \), \( n_1n_3 = 0 \), and all products of \( \sigma_1 \) and \( \sigma_2 \) with all other classes trivial except for the product \( \sigma_1n_1 = \sigma_2 \).

Most of the Steenrod squares are straightforward to calculate, but we do show one, \( Sq^1m_2 \):

\[
Sq^1m_2 = Sq^1(x_1y_1 + y_1^2, u_2) = (x_1^2y_1 + x_1y_1^2, \nu_3) = (x_1, 0)(x_1y_1 + y_1^2, u_2) + (0, \nu_3) = n_1m_2 + n_3.
\]

And the rest:

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<th>( \sigma_1 )</th>
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<th>( m_2 )</th>
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</table>

3.2. The Case \( \Gamma_6 \). Flöge’s presentation for \( \Gamma_6 \) is \( \langle A, B, M, R, S, U, W \rangle; \ A^2 = B^2 = M^2 = R^3 = S^3 = (BR)^3 = (BS)^3 = 1 \), \( AS = MR \), \( U^{-1}AU = M \), \( U^{-1}SU = R \), \( W^{-1}MW = A \), \( W^{-1}RBW = SB \) [7]. Let \( G \) be the subgroup generated by the group elements \( A, B, M, R, S \) and their relations. The following subgroups of \( G \),

\[
G_1 = \langle S, A, B; A^2 = S^3 = (BR)^3 = (BS)^3 = 1 \rangle \cong \mathbb{Z}/2 * A_4,
\]

\[
G_2 = \langle R, B, M; M^2 = R^3 = B^2 = (BR)^3 = 1 \rangle \cong \mathbb{Z}/2 * A_4,
\]

\[
H = \langle B, AS = MR; B^2 = 1 \rangle \cong \mathbb{Z}/2 * \mathbb{Z},
\]

combine to form \( G \cong G_1 * H \cong (\mathbb{Z}/2 * A_4)*(\mathbb{Z}/2 * Z) \). The associated subgroups of the HNN extensions, \( \langle A, S \rangle \) and \( \langle M, RB \rangle \), are both isomorphic to \( PSL_2(\mathbb{Z}) \). Set \( G_3 = HNN(U, G, PSL_2(\mathbb{Z}), PSL_2(\mathbb{Z})) \). Then \( \Gamma_6 = HNN(W, G_3, PSL_2(\mathbb{Z}), PSL_2(\mathbb{Z})) \). Thus the cohomology of \( \Gamma_6 \) is calculated in three steps.

Let \( x_1, y_1 \), and \( z_1 \) denote the duals in cohomology of the group elements \( A, M \) and \( B \) respectively, and let the copies of \( H^*(A_4) \) in \( G_1 \) and \( G_2 \) be generated by the classes \( u_2, v_3 \), and \( u_3 \) and \( p_2, q_3 \), and \( r_3 \). Finally let \( H^1(\mathbb{Z}) \) be generated by \( t_1 \).

By Theorem 2.3, \( H^*(G) \) fits into the long exact sequence

\[
0 \rightarrow H^0(\mathbb{G}) \xrightarrow{\partial} H^0(A_4 * \mathbb{Z}/2) \oplus H^0(A_4 * \mathbb{Z}/2) \xrightarrow{\alpha} H^0(\mathbb{Z}/2 * \mathbb{Z}) \xrightarrow{\delta} \cdots.
\]

The map \( \alpha \) is the difference of restrictions given by Theorem 2.2:

\[
\alpha(u_2) = \alpha(p_2) = z_1^2; \quad \alpha(u_3) = \alpha(r_3) = z_1^2; \quad \alpha(v_3) = \alpha(q_3) = \alpha(x_1) = \alpha(y_1) = 0.
\]

So \( \alpha \) is a surjection in degrees two and higher, with classes \( x_1, y_1, \bar{u}_2 = u_2 + p_2, \bar{v}_{31} = v_3, \bar{v}_{32} = q_3 \), and \( \bar{w}_3 = w_3 + r_3 \) in the kernel. There are also two classes in degree two that arise as images of \( \delta: \sigma_2 = \delta(z_1) \) and \( \tau_2 = \delta(t_1) \). All products with \( \tau_2 \) vanish, as this is the only class that originates from a torsion-free subgroup. Products with \( \sigma_2 \) also vanish, as in the case of \( \Gamma_2 \). The relations \( \bar{u}_2^2 + \bar{v}_{31}^2 + \bar{v}_{32}^2 + \bar{w}_3^2(\bar{v}_{31} + \bar{v}_{32}) = \bar{v}_{31}\bar{v}_{32} = 0 \) are straightforward to find and can be confirmed with a Poincaré series. We remove bars and rename \( v_{31} \) as \( v_3 \) and \( v_{32} \) as \( v_3 \). Summarizing,
Lemma 3.3. \( H^*(G) \cong \mathbb{F}_2[x_1, y_1, u_2, v_3, w_3](\sigma_2, \tau_2)/R, \) where \( R \) is generated by the relations \( u_3^2 + v_3^2 + \bar{v}_3^2 + w_3^2 + w_3(v_3 + \bar{v}_3) = x_1y_1 = 0, \) all products of \( x_1 \) and \( y_1 \) with \( u_2, v_3, \bar{v}_3, \) and \( w_3 \) trivial, and all products with \( \sigma_2 \) and \( \tau_2 \) trivial.

We add the group element \( U \) and calculate the cohomology of the first HNN extension, \( H^*(G_3) \). Recall that the associated subgroup, \( PSL_2(Z) \), is generated by \( A \) and \( S \). We only need consider \( U \)'s action on \( A \), as \( S \) is of order 3. Let \( u_1 \) be the element in \( H^*(G) \) dual to the group element \( A \), let \( i : PSL_2(Z) \to G_3 \) be the injection, and let \( U^{-1}AU = M \) induce the twisting. \( H^*(G_3) \) fits into the long exact sequence from Theorem 2.3,

\[
0 \to H^0(G_3) \overset{\partial}{\to} H^0(G) \overset{\partial}{\to} H^0(PSL_2(Z)) \overset{\delta}{\to} H^1(G_3) \overset{\partial}{\to} \cdots.
\]

The map \( \alpha \) is the difference of the restriction map \( i^* \) and the twisting map \( \theta^* \). As \( A \) generates one copy of \( Z/2 \) in \( G \), it follows that \( i^*(x_1) = \theta^*(y_1) = u_1, \) \( \alpha(y_1^n) = u_1 \), and \( \alpha \) is a surjection in degrees 1 and higher. The kernel of \( \alpha \) is generated by \( x_1 + y_1, u_2, v_3, \bar{v}_3, \) and \( w_3 \). The above long exact sequence breaks into short exact sequences,

\[
0 \to H^i(G_3) \overset{\beta}{\to} H^i(G) \overset{\alpha}{\to} H^i(PSL_2(Z)) \to 0.
\]

To simplify, let \( \bar{x}_1 = x_1 + y_1 \) and \( \sigma_1 = \delta(1) \).

Lemma 3.4. \( H^*(G_3) \cong \mathbb{F}_2[x_1, u_2, v_3, \bar{v}_3, w_3](\sigma_2, \tau_2)/R, \) where \( R \) is generated by the relations \( u_2^3 + v_3^2 + \bar{v}_3^2 + w_3^2 + w_3(v_3 + \bar{v}_3) = 0, \) all products with \( x_1, \sigma_1, \sigma_2 \) and \( \tau_2 \) trivial.

For \( \Gamma_6 \cong HNN(W, G_3, PSL_2(Z), PSL_2(Z)) \), let \( w_1 \) denote the dual of the group element \( M \) in \( H^1(PSL_2(Z)) \), where \( PSL_2(Z) \) is generated by \( M \) and \( RB \). Note that the cohomology classes \( x_1 \) and \( y_1 \) are dual to \( A \) and \( M \), and that neither \( A \) nor \( M \) is contained in a copy of \( A_4 \). As \( W^{-1}MW = A, \) \( i^*(y_1^i) = \theta^*(x_1^i) = w_1^i \). Thus \( \alpha \) is zero on \( x_1 \) and all other generators of \( H^*(G_3) \), yielding short exact sequences

\[
0 \to H^{i-1}(PSL_2(Z)) \overset{\delta}{\to} H^i(\Gamma_6) \overset{\beta}{\to} H^i(G_3) \to 0.
\]

This produces the exterior HNN class \( \delta(1) = \tau_1 \in H^1(\Gamma_6) \) and \( \delta(w_1^k) \in H^{k+1}(\Gamma_6) \) in higher degrees.

These classes generate new products that did not occur in the Bianchi group \( \Gamma_2 \). For \( i^*(y_1^i) = w_1^i \) implies that \( i^*(\bar{x}_1^k) = w_1^k \). Then by Proposition 2.7 \( \delta(w_1^k) = \delta(i^*(\bar{x}_1^k)) = \delta(i^*(\bar{x}_1^k) \cup 1) = \bar{x}_1^k \cup \delta(1) = \bar{x}_1^{k+1} \cup \tau_1 \). Proposition 2.7 also implies that all other products with \( \tau_1 \) are zero, as no other class is in the image of \( \delta \).

Theorem 3.5. \( H^*(\Gamma_6) \cong \mathbb{F}_2[\bar{x}_1, u_2, v_3, \bar{v}_3, w_3](\sigma_1, \tau_1, \sigma_2, \tau_2)/R, \) where \( R \) is generated by the relations \( u_2^3 + v_3^2 + \bar{v}_3^2 + w_3^2 + w_3(v_3 + \bar{v}_3) = 0, \) all products with \( \sigma_1, \tau_1, \sigma_2 \) and \( \tau_2 \) trivial, and all products with \( \bar{x}_1 \) and \( \tau_1 \) trivial except for \( \bar{x}_1^k \tau_1 \). Moreover,

\[
P(\Gamma_6, t) = \frac{-2t^5 - t^4 + 3t^3 + 2t - 1}{(1-t^2)(1-t^3)} + 2t + 2t^2.
\]

The Steenrod operations for \( H^*(\Gamma_6) \) are:
4. THE BIANCHI GROUPS $\Gamma_5$, $\Gamma_{10}$, $\Gamma_1$, $\Gamma_{11}$ AND $\Gamma_7$

These Bianchi groups use the same techniques as the Bianchi groups in the previous section. The calculations for the first two groups, non-Euclidean cases, are more complicated, so we include more details for the determination of their cohomology rings. The last three groups are comparatively simple; we only give a quick summary of the calculations.

4.1. The Case $\Gamma_5$. The presentation for $\Gamma_5$ is $\langle A, B, M, R, S, U, W; A^2 = B^2 = M^2 = R^3 = S^3 = (AB)^2 = (BM)^2 = 1, AS = MR, U^{-1}AU = M, U^{-1}SU = R, W^{-1}MBW = AB, W^{-1}RW = S \rangle$ [7]. $\Gamma_5$, like $\Gamma_6$, is a double HNN extension with base $G = \langle A, B, M, R, S; A^2 = B^2 = M^2 = R^3 = S^3 = (AB)^2 = (BM)^2 = 1, AS = MR \rangle$. Let

$$
G_1 = \langle S, A, B; S^3 = A^2 = B^2 = (AB)^2 = 1 \rangle \cong \mathbb{Z}/3 \ast \mathbb{D}_2,
$$

$$
G_2 = \langle R, B, M; R^3 = B^2 = M^2 = (MB)^2 = 1 \rangle \cong \mathbb{Z}/3 \ast \mathbb{D}_2,
$$

$$
H = \langle B, AS = MR, B^2 = 1 \rangle \cong \mathbb{Z}/2 \ast \mathbb{Z}.
$$

Let $G$ be the amalgamated product $G_1 \ast_H G_2 \cong (\mathbb{Z}/3 \ast \mathbb{D}_2) \ast (\mathbb{Z}/2 \ast \mathbb{Z})$. The twisted subgroups, $\langle A, S \rangle$ and $\langle MB, R \rangle$, are both isomorphic to $PSL_2(\mathbb{Z})$. Set $G_3 = HNN(U, G, PSL_2(\mathbb{Z})), PSL_2(\mathbb{Z}))$ and $\Gamma_5 = HNN(W, G_3, PSL_2(\mathbb{Z}), PSL_2(\mathbb{Z}))$.

As $H^*(\mathbb{Z}/3) \cong \mathbb{F}_2$, we refer to both $G_1$ and $G_2$ as $\mathbb{D}_2$ for brevity. Let $x_1, y_1, s_1$, and $t_1$ denote the duals in cohomology of the group elements $A, AB, MB$, and $M$ respectively; let $z_1$ denote the dual of $B$ in $H^1(H)$; and let $c_1$ generate $H^1(\mathbb{Z})$.

Then $H^*(G)$ fits into the long exact sequence

$$
0 \rightarrow H^0(G) \xrightarrow{\alpha} H^0(\mathbb{D}_2) \oplus H^0(\mathbb{D}_2) \xrightarrow{\alpha} H^0(\mathbb{Z}/2 \ast \mathbb{Z}) \xrightarrow{\delta} H^1(G) \xrightarrow{\delta} \cdots.
$$

Here $\alpha$ is the difference of restriction maps induced by the two inclusions $i_1 : H \rightarrow G_1$ and $i_2 : H \rightarrow G_2$. As $B = A \cdot AB = M \cdot MB$, it follows that $i_1^*(x_1) = i_1^*(y_1) = i_2^*(s_1) = i_2^*(t_1) = z_1$, and $\alpha_i(x_1)^i y_1^i = \alpha(s_1)^i t_1^i = z_1^{i+1}$. The map $\alpha$ is surjective except in degree one; the three classes $p_1 = (x_1 + y_1, 0), q_1 = (0, s_1 + t_1)$, and $r_1 = (x_1, t_1)$ generate the kernel of $\alpha$ as a ring up to nilpotent elements with one relation, $p_1 q_1 = 0$. We obtain an additional class, $\sigma_2 \in H^2(G)$, which is the image of $c_1$ under $\delta$, which is not in the image of $\alpha$ as it is the only cohomology class which does not originate from a finite subgroup. As in the case $\Gamma_2$, all products with this class are trivial.

**Lemma 4.1.** $H^*(G) \cong \mathbb{F}_2[p_1, q_1, r_1][\sigma_2]/R$, where $R$ is generated by the relations $p_1 q_1 = 0$ and all products with $2$ trivial.

For $G_3 \cong HNN(U, G, PSL_2(\mathbb{Z}), PSL_2(\mathbb{Z}))$, denote by $u_1$ the element in $H^1(PSL_2(\mathbb{Z}))$ dual to the group element $A$. We use the long exact sequence associated to the HNN extension

$$
0 \rightarrow H^0(G_3) \xrightarrow{\beta} H^0(G) \xrightarrow{\alpha} H^0(PSL_2(\mathbb{Z})) \xrightarrow{\delta} H^1(G_3) \xrightarrow{\delta} \cdots.
$$

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Let $\sigma_1 = \delta(1) \in H^1(G_3)$ be the exterior HNN class. From the injection $i : PSL_2(\mathbb{Z}) \to G$, $i^*$ sends $x_1$, the element dual to $A$ in $H^1(G)$, to $u_1$. The twisting part of $\alpha$ is induced by $U^{-1}AU = M$, so $\theta^*$ sends $t_1$ to $u_1$. This completely describes $\alpha$, but as in the case of $\Gamma_2$ we must take some care when we describe how $\alpha$ acts on products of generators of $H^*(G)$. We find that $\alpha(t_1) = \alpha(x_1, t_1) = u_1 + u_1 = 0$, but for products with $i > 0$ we have $\alpha(p_1^i r_1^i) = \alpha(q_1^i r_1^i) = u_1^{i+1}$.

The classes $l_1 = p_1 + q_1 = (x_1 + y_1, s_1 + t_1)$, $m_1 = r_1 = (x_1, t_1)$, and $m_2 = p_1^2 + p_1 r_1 = (y_1^2 + x_1 y_1, 0)$ generate the kernel of $\alpha$ up to nilpotence. They satisfy the relation $m_2^2 + l_1^2 m_2 + l_1 m_2 = 0$, and the standard argument shows that all products with $\sigma_1$ are trivial.

**Lemma 4.2.** $H^*(G_3) \cong \mathbb{F}_2[l_1, m_1, m_2](\sigma_1, \sigma_2)/R$, where $R$ consists of the relations $m_2^2 + l_1^2 m_2 + l_1 m_2 = 0$, and all products with $\sigma_1$ and $\sigma_2$ trivial.

By Theorem 2.3, $H^*(\Gamma_5)$ is calculated from

\[ 0 \to H^0(\Gamma_5) \overset{\beta}{\to} H^0(G_3) \overset{\alpha}{\to} H^0(PSL_2(\mathbb{Z})) \overset{\delta}{\to} H^1(\Gamma_5) \overset{\beta}{\to} \cdots. \]

Let $\tau_1 = \delta(1) \in H^1(\Gamma_5)$ be the exterior HNN class. Also let $w_1 \in H^1(PSL_2(\mathbb{Z}))$ denote the dual of the element $MB$ and recall that $W$ sends $MB$ to $AB$, and $R$ to $S$. We only need consider the first two elements, as the latter two are of order three. From the inclusion $i : PSL_2(\mathbb{Z}) \to G_3$ and the twisting of $W$, we have that $i^*(s_1) = \theta^*(y_1) = w_1$. Thus $\alpha(s_1) = \alpha(y_1) = w_1$, and $\alpha(t_1) = \alpha(x_1) = 0$.

In particular, $\alpha(m_1) = \alpha(x_1, t_1) = 0$, and the image under $\alpha$ of any product in $H^*(G_3)$ containing $m_1$ will also be zero, as its first component will be divisible by $x_1$ and its second by $t_1$. For other products,

\[
\alpha(l_1^i) = \alpha((x_1 + y_1)^i, (s_1 + t_1)^i) = w_1^i + w_1^i = 0,
\]

\[
\alpha(l_1^i m_2^j) = \alpha((x_1 + y_1)^i(y_1^2 + x_1 y_1)^j, 0) = w_1^{i+2j}.
\]

The map $\alpha$ is a surjection except in degree one, which yields the class $\tau_2 = \delta(w_1) \in H^2(\Gamma_5)$. The three classes $l_1, m_1$, and $m_3 = m_1 m_2 = (x_1^2 y_1 + x_1 y_1^2, 0)$ are in the kernel of $\alpha$; they satisfy the relation $m_3(m_3 + l_1^2 m_1 + l_1 m_2^2) = 0$. By dimensional arguments similar to the case $\Gamma_2$, all products with $\tau_1$ and $\tau_2$ are zero.

**Theorem 4.3.** $H^*(\Gamma_5) \cong \mathbb{F}_2[l_1, m_1, m_3](\sigma_1, \sigma_2, \tau_1, \tau_2)/R$, where $R$ is generated by the relations $m_3(m_3 + l_1^2 m_1 + l_1 m_2^2) = 0$, and all products of exterior classes with other classes trivial. Moreover,

\[
P(\Gamma_5, t) = \frac{1 + t^3}{(1 - t)^2} + 2t + 2t^2.
\]

The Steenrod operations:

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<th>$Sq^1$</th>
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<th>$m_1^1$</th>
<th>$m_3^1$</th>
<th>$\sigma_1$</th>
<th>$\sigma_2$</th>
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4.2. The Case $\Gamma_{10}$. In many regards this is the most complicated of the groups we have considered so far, as it is a triple HNN extension. However, no new techniques are required for the calculations. As in the other cases, we build up this group in stages. Flöge gives the presentation of $\Gamma_{10}$ as $\langle A, B, L, S, D, U, W; \quad A^2 = B^2 = L^2 = S^3 = (AB)^2 = (AL)^2 = 1 \rangle$ with other relations involving $D$, $U$, and $W$ that we give in their respective extensions [7]. The base group $G_0$ has the presentation $\langle A, B, L, S; \quad A^2 = B^2 = L^2 = S^3 = (AB)^2 = (AL)^2 = 1 \rangle$, which we break into pieces:

$$G_{01} = \langle A, B; A^2 = (AB)^2 = B^2 = 1 \rangle \cong D_2,$$

$$G_{02} = \langle A, L; A^2 = (AL)^2 = L^2 = 1 \rangle \cong D_2,$$

$$G_{03} = \langle S \rangle \cong \mathbb{Z}/3,$$

$$H = \langle A \rangle \cong \mathbb{Z}/2.$$

With this decomposition,

$$G_0 \cong (G_{01} *_H G_{02}) * G_{03} \cong (D_2 *_{\mathbb{Z}/2} D_2) * \mathbb{Z}/3.$$

The first HNN extension, $G_1$, adds the element $D$. Its presentation is $\langle G_0, D; D^{-1}ALSD = S^{-1}AB \rangle$, with $\langle ALS \rangle \cong \mathbb{Z}$. $G_2$, the second HNN extension, adds the group element $U$. In particular,

$$G_2 = \langle G_1, U; U^{-1}DABD^{-1}U = D^{-1}ALD, U^{-1}LDS^{-1}D^{-1}U = BD^{-1}S^{-1}D \rangle.$$

The subgroup $\langle DABD^{-1}, LDS^{-1}D^{-1} \rangle$ is isomorphic to $\mathbb{Z}/2 * \mathbb{Z}$. The final HNN extension is

$$\Gamma_{10} = \langle G_2, W; W^{-1}BW = U^{-1}LU, W^{-1}D^{-1}SDW = U^{-1}DS^{-1}U \rangle,$$

where $\langle B, D^{-1}SD \rangle$ is isomorphic to $PSL_2(\mathbb{Z})$.

The calculations for $H^*(G_0)$ are almost identical to those in the first stage of the calculation of $H^*(\Gamma_5)$, as $G_{03}$ is not detected with $\mathbb{F}_2$ coefficients. Let $s_1$ and $t_1$ be the duals in cohomology of the group elements $A$ and $B$ in $G_{01}$, let $x_1$ and $y_1$ be the duals of the group elements $A$ and $L$ in $G_{02}$, and let $\bar{z}_1$ be the dual of the group element $A$ in $H$. In the long exact sequence of Theorem 2.3, $\alpha$ is a surjection with kernel generated by $\bar{x}_1 = (x_1, s_1)$, $\bar{y}_1 = (y_1, 0)$, and $\bar{t}_1 = (0, t_1)$. These classes satisfy the relation $y_1t_1 = 0$.

Now $G_1$ is an HNN extension of $G_0$ where the element $D$ twists a subgroup isomorphic to $\mathbb{Z}$. From the long exact sequence

$$0 \rightarrow H^0(G_1) \xrightarrow{\delta} H^0(G_0) \cong H^0(\mathbb{Z}) \xrightarrow{\delta} H^0(G_1) \xrightarrow{\delta} \cdots$$

we see that the cohomology of $G_1$ is essentially the same as the cohomology of $G_0$, except for two new classes in $H^*(G_1)$: the HNN class $\sigma_1 = \delta(1)$, and $\sigma_2 = \delta(c_1)$ where $c_1 \in H^1(\mathbb{Z})$. All non-trivial products with these classes must lie in the image of $\delta$ by naturality, so the only possible relation is $\sigma_1^2 = \sigma_2$. But $\sigma_1$ is the HNN class, so $\sigma_1^2 = 0$.

**Lemma 4.4.** $H^*(G_1) \cong \mathbb{F}_2[x_1, t_1, y_1](\sigma_1, \sigma_2)/R$, where $R$ is generated by the relations $yt = 0$, and all products with $\sigma_1$ and $\sigma_2$ zero.

The calculations for $H^*(G_2)$ are more involved. In the associated subgroup, $\langle DABD^{-1}, LDS^{-1}D^{-1} \rangle \cong \mathbb{Z}/2 * \mathbb{Z}$, we let $e_1$ denote the class in $H^1(\mathbb{Z})$, and let $c_1$ be the polynomial generator in $H^1(\mathbb{Z}/2)$. Since conjugation induces the identity map in cohomology, $AB$ and $DABD^{-1}$ represent the same cohomology class. In
particular, without loss of generality, $c_1$ is dual to either $AB$ or $DABD^{-1}$. A similar argument applies to the two group elements $AL$ and $D^{-1}ALD$. Consider the long exact sequence

$$0 \to H^0(G_2) \xrightarrow{\beta} H^0(G_1) \xrightarrow{\alpha} H^0(\mathbb{Z}/2 \ast \mathbb{Z}) \xrightarrow{\delta} H^1(G_2) \xrightarrow{\beta} \ldots.$$

From the injection $i : \mathbb{Z}/2 \ast \mathbb{Z} \to G_1$, $i^*$ sends both $s_1$ and $t_1$ to $c_1$, as $AB = A \cdot B$; thus $i^*(x_1) = i^*(t_1) = c_1$. Similarly, the twisting sends both $x_1$ and $y_1$ to $c_1$, so $\theta^*(x_1) = \theta^*(y_1) = c_1$. It is then easy to verify that $\alpha(x_1^2) = 0$ and $\alpha(x_1^2t_1^k) = \alpha(x_1^2y_1^k) = c_1^{2+k}$, $k > 0$.

The classes $m_1 = x_1 = (x_1, s_1)$, $l_1 = y_1 + l_1 = (y_1, t_1)$, and $m_2 = x_1y_1 + y_1^2 = (x_1y_1 + y_1^2, 0)$ generate most of the kernel of $\alpha$. As in the case $\Gamma_5$, these classes satisfy the relation $m_2^2 + l_1^2m_2 + l_1m_1m_2 = 0$. Two other classes come from the boundary operator, $\tau_1 = \delta(1) \in H^1(G_2)$ and $\tau_2 = \delta(x_1) \in H^2(G_2)$. By similar arguments as before, products with these classes are trivial.

**Lemma 4.5.** $H^*(G_2) \cong \mathbb{F}_2[l_1, m_1, m_2][\sigma_1, \sigma_2, \tau_1, \tau_2]/R$, where $R$ is generated by the relations $m_2^3 + l_1^2m_2 + l_1m_1m_2 = 0$, and all products with the four exterior classes trivial.

$\Gamma_{10}$ is the final HNN extension. Recall that the associated subgroup is

$$(B, D^{-1}SD) \cong PSL_2(\mathbb{Z}).$$

The twisting, $\theta$, in the HNN extension sends $B$ to $U^{-1}LU$, which represents the same cohomology class as $L$. This situation is identical to the calculations in the final stage of $\Gamma_5$. We refer the reader there, and state the result.

**Theorem 4.6.** $H^*(\Gamma_{10}) \cong \mathbb{F}_2[l_1, m_1, m_3][\sigma_1, \sigma_2, \tau_1, \tau_2, \eta_1, \eta_2]/R$, where $R$ is generated by the relations $m_3(m_3 + l_1^2m_1 + l_1m_1^2) = 0$, and all products of exterior classes with other classes trivial. Moreover,

$$P(\Gamma_{10}, t) = \frac{1 + t^3}{(1 - t)^2} + 3t + 3t^2.$$

Not only does this ring closely match $H^*(\Gamma_5)$, but the Steenrod squares are identical.

### 4.3. The Case $\Gamma_1$.

The group presentation for $\Gamma_1$ is $(A, B, C, D; A^3 = B^2 = C^3 = D^2 = (AC)^2 = (AD)^2 = (BD)^2 = (BC)^2 = 1)$ ([6], §4.4). Consider the subgroups

- $G_{11} = (A, C; A^3 = C^3 = (AC)^2 = 1) \cong A_4,$
- $G_{12} = (A, D; A^3 = D^2 = (AD)^2 = 1) \cong S_3,$
- $G_{21} = (B, C; B^2 = C^3 = (BC)^2 = 1) \cong S_3,$
- $G_{22} = (B, D; B^2 = D^2 = (BD)^2 = 1) \cong D_2.$

The Bianchi group $\Gamma_1$ is the amalgamated product $G_1 \ast_H G_2$, where $G_1$ and $G_2$ are themselves amalgamated products and $H$ is the modular group:

- $G_1 = G_{11} \ast_H G_{12}, \quad (A) \cong \mathbb{Z}/3,$
- $G_2 = G_{21} \ast_H G_{22}, \quad (B) \cong \mathbb{Z}/2,$
- $H = (C, D) \cong \mathbb{Z}/2 \ast \mathbb{Z}/3 \cong PSL_2(\mathbb{Z}).$

It is elementary to verify that if the amalgamated subgroup is cohomologically trivial, then Theorem 2.8 holds and the cohomology of the group is the reduced
the confirmed similar twisting, Theorem w1 element trivial. The
$q3$ as the sum so 4598 and $A2$ This equal $H*(D2)$. For the final amalgamation, one finds that $H*(\Gamma_1) \cong H^*(A_4) \oplus H^*(D_2)$. This isomorphism also extends to cup products and Steenrod squares.

4.4. The Case $\Gamma_{11}$. We use Fine’s presentation, $\Gamma_{11} = \langle A, T, U; (U^{-1}AU)^3 = A^2 = (AT)^3 = [T, U] = 1 \rangle$ (6), §4.3. Set $S = AT$, $W = AU$, $M = W^{-1}AW$, and $V = W^{-1}SW$. Replace the relation $(U^{-1}AU)^3 = 1$ with the relation $(ATU^{-1}AU)^3 = 1$. Note that

$$AV = AW^{-1}SW = AU^{-1}ASW = ATT^{-1}U^{-1}AATW$$

$$ST^{-1}U^{-1}TW = SU^{-1}W = SW^{-1}AW = SM.$$ 

The new presentation for $\Gamma_{11}$ is $\langle A, V, S, M, W; A^2 = S^3 = (AV)^3 = V^3 = M^2 = 1, AV = SM, W^{-1}AW = M, W^{-1}SW = V \rangle$. Set

$$G = \langle A, V, S, M; A^2 = S^3 = (AV)^3 = V^3 = M^2 = 1, AV = SM \rangle,$$

$$G_1 = \langle A, V; A^2 = V^3 = (AV)^3 = 1 \rangle \cong A_4,$$

$$G_2 = \langle S, M; M^2 = S^3 = (SM)^3 = 1 \rangle \cong A_4,$$

$$H = \langle AV = SM, (AV)^3 = 1 \rangle \cong \mathbb{Z}/3.$$ 

We can write $G$ as the amalgamated product $G_1 \ast_H G_2 \cong A_4 \ast_{\mathbb{Z}/3} A_4$. Then $\Gamma_{11} = HNN(W, G, PSL_2(\mathbb{Z}), PSL_2(\mathbb{Z}))$, with $PSL_2(\mathbb{Z}) = \langle A, S \rangle$.

The cohomology of the base group $G$ is directly calculated using Theorem 2.8, as $\mathbb{Z}/3$ has trivial cohomology in $\mathbb{F}_2$ coefficients. This yields $H^*(G) \cong \mathbb{F}_2[u_2, q_2, v_3, s_3]/R$, where $R$ is the set of relations $u_2^3 + v_3^3 + w_3^3 = 0$, $q_2^3 + r_3^3 + s_3^3 = 0$, and all products of $u_2, v_3, w_3$ with $q_2, r_3, s_3$, and so trivial.

To find $\Gamma_{11}$, we fit this information into the long exact sequence

$$0 \rightarrow H^0(\Gamma_{11}) \xrightarrow{\beta} H^0(G) \xrightarrow{\alpha} H^0(PSL_2(\mathbb{Z})) \xrightarrow{\delta} \cdots.$$ 

The map $\alpha$ is the difference of restriction and twisting maps. Notice that the group element $A$ is an element of order two in both $PSL_2(\mathbb{Z})$ and $G_1$, and that $W$’s twisting, $W^{-1}AW = M$, sends an element of $PSL_2(\mathbb{Z})$ to an element of $G_2$. Let $w_1$ denote $A$’s dual in $H^1(PSL_2(\mathbb{Z}))$. Then

$$i^*(u_2) = \theta^*(q_2) = w_1^2, \quad i^*(v_3) = \theta^*(r_3) = 0, \quad i^*(w_3) = \theta^*(s_3) = w_1^3.$$ 

The map $\alpha$ is a surjection in degrees two and higher, with classes $\bar{u}_2 = (u_2, q_2)$, $\bar{v}_3 = (v_3, 0)$, $\bar{w}_3 = (w_3, 0)$, and $\bar{s}_3 = (s_3, 0)$ in the kernel. We get two other classes, the HNN class $\delta(1) \in H^1(\Gamma_{11})$, and $\delta(w_1) \in H^2(\Gamma_{11})$ which we call $\sigma_1$ and $\sigma_2$. By similar arguments as in the case $\Gamma_2$, all products with these classes are zero. The relations $\bar{u}_2^3 + \bar{v}_3^2 + \bar{w}_3^2 + \bar{v}_3^2 = 0$ are easy to find, and can be confirmed with a Poincaré series.

The reader may wish to compare these calculations with the case $\Gamma_6$, which has similar classes arising from an amalgamated product. For readability, we remove the bars from all classes, rename $v_31$ by $v_3$, and rename $v_32$ by $\bar{v}_3$. We do not include the Steenrod squares, as these are identical to the case $\Gamma_6$.

**Theorem 4.7.** $H^*(\Gamma_{11}) \cong \mathbb{F}_2[u_2, v_3, \bar{v}_3, w_3](\sigma_1, \sigma_2)/R$, where $R$ is generated by the set of relations $u_2^3 + v_3^3 + \bar{v}_3^2 + w_1(v_3 + \bar{v}_3) = 0, v_3\bar{v}_3 = 0, and all products
with \(\sigma_1\) and \(\sigma_2\) trivial. Moreover,
\[
P(\Gamma_{11}, t) = \frac{t^6 + 2t^3 + 1}{(1 - t^2)(1 - t^3)} + t + t^2.
\]

4.5. The Case \(\Gamma_7\). Fine's presentation for \(\Gamma_7\) is \(\langle A, T, U; (U^{-1}AU)^2 = A^2 = (AT)^3 = [T, U] = 1 \rangle\) ([6], §4.3). This differs from \(\Gamma_{11}\)'s presentation by only the exponent of \(U^{-1}AU\); analogous transformations result in the following presentation for the group: \(\langle A, V, S, M, W; A^2 = S^3 = (AV)^2 = V^3 = M^2 = 1, AV = SM, W^{-1}AW = M, W^{-1}SW = V \rangle\). Set
\[
G = \langle A, V, S, M; A^2 = S^3 = (AV)^2 = V^3 = M^2 = 1, AV = SM \rangle,
\]
\[
G_1 = \langle A, V; A^2 = V^3 = 1 \rangle \cong S_3,
\]
\[
G_2 = \langle S, M; M^2 = S^3 = 1 \rangle \cong S_3,
\]
\[
H = \langle AV = SM, (AV)^2 = 1 \rangle \cong \mathbb{Z}/2.
\]

As before, \(G \cong \mathbb{S}_3 \ast_{\mathbb{Z}/2} \mathbb{S}_3\) and \(\Gamma_7 = HNN(W, G, \mathbb{P}SL_2(\mathbb{Z}), \mathbb{P}SL_2(\mathbb{Z}))\), where \(\mathbb{P}SL_2(\mathbb{Z}) = \langle A, S \rangle\).

This case is similar to, but easier than, the case \(\Gamma_{11}\). Briefly, one finds that \(H^*(G) \cong \mathbb{Z}/2\), and that the map \(i^*: H^*(\Gamma_7) \to H^*(\mathbb{P}SL_2(\mathbb{Z}))\) is non-zero.

**Theorem 4.8.** \(H^*(\Gamma_7) \cong \mathbb{F}_2[y_1](x_1)\), and \(P(\Gamma_7, t) = \frac{1 + t}{1 - t}\).

The Steenrod squares on the two generators are:

<table>
<thead>
<tr>
<th></th>
<th>(x_1)</th>
<th>(y_1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Sq^1)</td>
<td>0</td>
<td>(y_1^2)</td>
</tr>
</tbody>
</table>

5. The Case \(\Gamma_3\).

We use Flöge's presentation for the group, \(\Gamma_3 = \langle A, L, K, S, M; A^2 = L^3 = (AL)^2 = K^3 = S^3 = (KS)^2 = M^2 = (MS)^3 = 1, AL = K^2S^2, AL^2 = M \rangle\) [7]. We break this into subgroups,
\[
G_1 = \langle A, L; A^2 = L^3 = (AL)^2 = 1 \rangle \cong S_3,
\]
\[
G_2 = \langle K, S; K^3 = S^3 = (K^2S)^2 = 1 \rangle \cong A_4,
\]
\[
G_3 = \langle M, S; M^2 = S^3 = (MS)^3 = 1 \rangle \cong A_4,
\]
\[
H_1 = \langle AL = K^2S^2 \rangle \cong \mathbb{Z}/2,
\]
\[
H_2 = \langle S \rangle \cong \mathbb{Z}/3,
\]
\[
H_3 = \langle AL^2 = M \rangle \cong \mathbb{Z}/2.
\]

These subgroups fit into a "triangular product," which is related to an amalgamated product. Optimistically, as in the other cases, this group would act on a tree with a cyclic quotient and the above groups as isotropy. Unfortunately, we have the following result, due to Serre.

**Theorem 5.1 ([11]).** Say \(\Gamma_3\) acts on a tree \(X\). Then \(X^{\Gamma_3} \neq \emptyset\), where \(X^G\) represents the fixed points of \(X\) under the action of \(G\).
As a corollary, the quotient graph of any tree on which $\Gamma_3$ acts will contain at least one cell which has $\Gamma_3$ as an isotropy group. Clearly this quotient will be unusable for cohomology calculations. Fortunately, there is another option: the Mendoza complex mentioned in the first section. In his thesis, Mendoza shows, for any Euclidean Bianchi group $\Gamma_d$, how to construct a 2-dimensional deformation retract of $\mathbb{H}^3$ on which $\Gamma_d$ acts with finite isotropy and compact quotient. He then explicitly calculates this region for the five Euclidean cases. For $\Gamma_3$, the quotient is the compact cellular complex with isotropy shown in Figure 2 [9].

The isotropy of the 2-cell is the identity element. To relate the cohomology groups we need a gadget to take the place of the long exact sequence used in the other examples. This is the equivariant spectral sequence. Details are in Brown’s book [5], but we quickly summarize them here.

For a group $G$, say $X$ is a cellular space on which $G$ acts and let $C^*(X)$ be the cellular cochain complex. Say also that $F$ is a projective resolution of $\mathbb{F}_2$ over $\mathbb{F}_2G$. We define $H^*(G, C^*(X))$ to be $H^*(\text{Hom}_{\mathbb{F}_2}(F, C^*(X)))$. These are known as the equivariant cohomology groups of $X$, denoted by $H_G^*(X)$. Notice that if $X$ is contractible, as in our case, then $H_G^*(X) \cong H^*(G)$. From analysis of the horizontal and vertical filtrations of the double complex $\text{Hom}_{\mathbb{F}_2}(F, C^*(X))$ we get a spectral sequence with $E^{p,q}_2 = H^q(G, C_p^q(X))$ which converges to $H^{p+q}(G)$. By Shapiro’s lemma, we can also identify $E^{p,q}_2$ with $\prod_{\sigma \in \Sigma_p} H^q(G_\sigma)$, where $\Sigma_p$ is a set of representatives of $p$-cells in $X/G$.

This spectral sequence has a number of desirable properties. First, the spectral sequence has a product $E^{p,q}_2 \otimes E^{s,t}_2 \to E^{p+s,q+t}_2$ which is compatible with the standard cup product on $H^*(G)$. Second, the products in $E^{0,*}_2$, the vertical edge, are compatible with the products in $\prod_{\sigma \in \Sigma_p} H^q(G_\sigma)$. Third, the differential $d_1$ is a difference of restriction and twisting maps based on inclusion and identification in the quotient complex $X/G$. When this quotient is a line segment or a single edge and vertex, the spectral sequence consists of only two columns and collapses at the $E_2$ term to give a Wang sequence that is the long exact sequence we had before. In this case the differential $d_1$ is the same as the map $\alpha$.

In the calculation of $H^*(\Gamma_3)$ we have $E^{p,q}_1 = 0$ for $p > 1$, with the exception of $E^{2,0}_1 \cong \mathbb{F}_2$. Thus in $E_2$ term of the spectral sequence in Figure 3, showing the differentials and cohomology groups in a generic row.

Let $H^*(G_1) \cong \mathbb{F}_2[x_1]$, $H^*(G_2) \cong \mathbb{F}_2[u_2, v_3, w_3]/R$, and $H^*(G_3) \cong \mathbb{F}_2[q_2, r_3, s_3]/R$ (where $R$ are the relations for $H^*(A_4)$). Likewise, let $H^*(H_1) \cong \mathbb{F}_2[y_1]$, $H^*(H_2) \cong \mathbb{F}_2$, and $H^*(H_3) \cong \mathbb{F}_2[1]$. At the zero level, let $\eta_1, \eta_2$, and $\eta_3$ generate $H^0(G_1)$, let $\mu_1, \mu_2$, and $\mu_3$ generate $H^0(H_1)$, and let $\nu$ generate $H^0(\{1\})$. Notice that the elements $AL$ and $AL^2$ in $S_3$ are conjugate. Thus they represent the same cohomology
class, \(x_1\), in \(H^1(G_1)\). In the previous cases \(\alpha\) was the key map in understanding the cohomology, so it is no surprise that here we want to understand the differential \(d_1\). From the quotient diagram in Figure 2 we see that no edges are identified, so \(d_1\) consists only of restriction maps. The inclusion of the isotropy of the edges into the isotropy of the vertices determines \(d_1\)'s action on the cohomology generators:

\[
d_1(\eta_1) = \mu_1 + \mu_3, \quad d_1(\eta_2) = \mu_1 + \mu_2, \quad d_1(\eta_3) = \mu_2 + \mu_3, \\
d_1(\mu_1) = d_1(\mu_2) = d_1(\mu_3) = \nu, \\
d_1(u_2) = y_1^2, \quad d_1(v_3) = 0, \quad d_1(w_3) = y_1^3, \\
d_1(q_2) = z_1^2, \quad d_1(r_3) = 0, \quad d_1(s_3) = z_1^3, \\
d_1(x_1) = y_1 + z_1, \quad d_1(\nu) = 0.
\]

So \(d_1 : E_1^{0,p} \rightarrow E_1^{1,p}\) is a surjection for \(p \geq 2\). At \(E_1^{0,0}\), \(\dim_{\mathbb{F}_2} \ker(d_1) = 1\) (generated by \(\eta_1 + \eta_2 + \eta_3\)), and \(\dim_{\mathbb{F}_2} \im(d_1) = 2\). At \(E_1^{1,0}\), \(\dim_{\mathbb{F}_2} \ker(d_1) = 2\) (generated by \(\mu_1 + \mu_2\) and \(\mu_2 + \mu_3\)), and \(\dim_{\mathbb{F}_2} \im(d_1) = 1\). At \(E_1^{0,2}\), \(\dim_{\mathbb{F}_2} \ker(d_1) = 1\). Thus \(\dim_{\mathbb{F}_2}(E_2^{0,0}) = 1\), and nothing else from that row survives to \(E_\infty\). Also, note that \(d_1\) is not onto at \(E_1^{1,1}\).

On the \(E_2\) page, there are no non-zero classes in the second column. Therefore \(d_2 = d_\infty = 0\), and \(E_2 \cong E_\infty\). From the first row up, all classes in the first column are in the image of \(d_1\), with the exception of one class in \(E_1^{1,1}\). This class, which we call \(\sigma_2\), will survive to \(E_\infty\). In the zero column, there are a number of classes which are in the kernel of \(d_1\). In particular \((x_1^2, u_2, q_2), (x_1^3, w_3, s_3), (0, v_3, 0), \) and \((0, 0, r_3)\) and the ideal generated by these classes all survive to the \(E_\infty\) page. Call these classes \(\bar{u}_2, \bar{w}_3, \bar{v}_{31}, \) and \(\bar{v}_{32}\) respectively. By the cup product compatibility of this vertical edge, these classes satisfy the relations

\[
\bar{u}_2 v_3^3 + \bar{w}_3 v_3^2 + \bar{v}_{31} v_3^2 + \bar{v}_{32} (\bar{v}_{31} + \bar{v}_{32}) = 0, \quad \bar{v}_{31} \bar{v}_{32} = 0.
\]

We also need to determine all products with the class \(\sigma_2\). This class lies in \(E_2^{1,1}\), and, by the compatibility of the spectral sequence with the cup product, any product with this element must lie in the first column or to its right. But \(\sigma_2\) is the only class in this column, and so it must multiply trivially with all other classes, including itself. We remove the bars for brevity, and as in the case \(\Gamma_6\) rename the class \(\bar{v}_{31}\) as \(v_3\) and the class \(\bar{v}_{32}\) as \(v_3\). The Poincaré series for \(H^*(\Gamma_3)\) is almost
identical to the case $\Gamma_{11}$, as are the final ring structure and the Steenrod squares. We refer the reader to the details of that case, and simply state the result.

**Theorem 5.2.** $H^*(\Gamma_3) \cong \mathbb{F}_2[v_2, v_3, \tilde{v}_3, \sigma_2]/R$, where $R$ is generated by the relations $u_2^3 + u_3^3 + v_3^3 + w_3 + 3v_3\tilde{v}_3 = 0$, $v_3\tilde{v}_3 = 0$, and all products with $\sigma_2$ trivial. Moreover,

$$P(\Gamma_3, t) = \frac{t^6 + 2t^3 + 1}{(1 - t^2)(1 - t^3)} + t^2.$$

**Remark 5.3.** Upon review, there are many similarities between the cohomology rings calculated above. The cases $\Gamma_5$ and $\Gamma_{10}$ are a close match, as are the cases $\Gamma_6, \Gamma_{11}$, and $\Gamma_3$, despite their very different group presentations. This is probably a reflection of how the Bianchi groups are fit together from their finite subgroups; there are just so many possible ways to combine them. Of course, a sample of eight examples is a bit sparse. It would be interesting to try some other cases and see how their cohomology rings compare to the known examples. The hard part is building the proper group presentations.

The Poincaré series provides a check on previous calculations of the homology of the Bianchi groups. Using the universal coefficient theorem, one can relate homology with $\mathbb{Z}$ coefficient to our cohomology calculations. This shows that there are a few inaccuracies in [10]. In particular, the results for $H^*(\Gamma_1)$ are short a copy of $\mathbb{Z}/2$ in dimensions 5 and 9 mod 12; and for the case $\Gamma_3$ there are two copies too many in dimensions 6 mod 12.

**References**


**Department of Mathematics, Lafayette College, Easton, Pennsylvania 18042**  
**E-mail address:** berkovee@lafayette.edu