Logic as a Matrix for Bierce’s Thought: “The Gem Puzzle”

One of the most unusual and atypical non-fictional articles Ambrose Bierce ever wrote appeared in the 22 May 1880 issue of the Argonaut, a San Francisco literary journal. Titled simply “The Gem Puzzle,” it occupies almost half a page and is wholly devoted to the analysis and solution of the mathematical novelty variously known as the “Gem Puzzle” or the “Fifteen Puzzle.” There is no doubt that Bierce was the author, for the initial “B.” appears at its end, which Bierce used distinctively when he did not sign his name. Bierce apparently never returned to the subject, nor did he ever again write an article with such a level of technicality.

What keeps “The Gem Puzzle” article from being a mere curiosity is its significance as a demonstration of the analytical and organizational powers of Bierce’s mind. Although his fame justly rests on his short fiction, particularly his stories of the Civil War and the wickedly cynical definitions of his Devil’s Dictionary, in fact the greatest bulk of his writings over the course of more than forty years he spent as a journalist consists of the largely uncollected columns of his running commentary on local, national, and international events. Under different names (“The Town Crier,” “Comments, Mostly Frivolous, on the Fad of the Day,” “The Passing Show,” “The Bald Campaigner,” but most extensively and famously “Prattle: A Transient Record of Individual Opinion”) Bierce’s feature columns appeared prominently once or twice a week in the San Francisco Examiner and in various other periodicals (e.g., newspapers owned by the Hearst syndicate) which published his work. Although Bierce tended to be self-deprecating about his comments and opinions, his uncollected columns contain some of the
most thoughtful and incisive analyses of current events in American journalism.

At the time of the composition of “The Gem Puzzle,” Bierce was still editing as well as writing for the *Argonaut*. It is not surprising for the *Argonaut* to have printed the article for, in addition to Bierce’s role in deciding what would go into the journal, the Gem Puzzle was an enormously popular fad that was then sweeping the country, and so the topic was of certain interest. But it is somewhat surprising that Bierce undertook to write about it himself. His formal education after public school consisted of his attendance, at least in 1859 when he was seventeen, at the Kentucky Military Institute. Hitherto, the main influence of that school has always been assumed to be training in topographical engineering, for once Bierce received a commission in the Union army, he was assigned to prepare maps in advance of battles. With the authorship of this article, however, it is now apparent that Bierce had also acquired some significant, though not professional, mathematical skills. The concise but rather dense style of this article also suggests the model of a formal academic paper. If so, it was a reversion to an early influence that he had left behind when he adopted a brisker expository style more appealing to readers.

The Gem Puzzle that Bierce analyzed remains widely recognizable today as one of the so-called sliding block puzzles encountered during one’s childhood. The version Bierce probably used consisted of fifteen removable blocks labeled 1 through 15 that were slid around on a $4 \times 4$ grid. According to the directions on the box, one was to “[p]lace the Blocks in the Box irregularly, then move until in regular order.” What made the puzzle so maddening—and interesting—was that for certain “irregular placements of the blocks” it seemed impossible to move the blocks back to the regular order shown in Figure 1 below. Of particular note, starting from the solution, it seemed impossible to slide the blocks around in such a way that in the end blocks 1 through 13 were in their original positions but blocks 14 and 15 were swapped (see the right side of Figure 1). That this task really is hopeless can be rigorously proved using mathematics, although the arguments required are quite involved.

The Gem Puzzle may seem pedestrian by today’s standards, but it was a sensation when it made its appearance in Boston in December 1879 (*15 Puzzle* 11). Like the Rubik’s Cube craze that occurred almost exactly one hundred years later, the Gem Puzzle spread coast-to-coast, quickly becoming a public mania. In March 1880, the *New York Times* published a satirical editorial describing how the puzzle had spread all the way to the White House:
No pestilence has ever visited this or any other country which has spread with the awful celerity of what is called the “Fifteen Puzzle.” It has spread over the entire country. Nothing arrests it. . . . Who introduced the Fifteen Puzzle into the White House no one knows, but in all probability the guilty person was a Southern Brigadier of more than usual villainy.

Although the Gem Puzzle loosened its grip on the public by mid-year 1880, it generated a legacy of sliding puzzles that continues to this day. It remains an important archetype whose background and history is of such interest that the Gem Puzzle is the subject of a recent fascinating book by Jerry Slocum and Dic Sonneveld titled _The 15 Puzzle_.

Bierce may have first been introduced to the puzzle on 9 March 1880 when “the San Francisco Chronicle republished a comprehensive article about the Fifteen Puzzle, titled, ‘The Latest Craze’ from the Philadelphia Times” (15 Puzzle 50). Certainly, the detail in his article suggests that he spent considerable time working on the puzzle, studying its behavior and looking for patterns. Furthermore, Bierce must have considered his investigations interesting and unusual enough to warrant devoting an entire article to it. Bierce was one of many people whose experiences with the puzzle appeared in print. However, there are two main features that sets Bierce’s article apart. The first is the depth and breadth of his analysis, which is unusually complete. Many attempted solutions addressed solely the problem of rearranging the blocks by sliding so that only blocks 14 and 15 were swapped. Bierce, on the other hand, wanted a more general solution, and strove to distinguish rearrangements of the fifteen blocks which could be realized by sliding from those which could not. Indeed, Bierce makes the statement early in his article that he will show “what [configurations] of the fifteen numbers can and cannot be reduced to order.” In this regard, he really approached the problem the way a mathematician might, trying
to understand the puzzle’s entire structure, rather than only focusing on a single aspect of it. The second feature is Bierce’s overall approach to his analysis, which comes in two stages. He first analyzes how blocks can be rearranged in the bottom row of the Gem Puzzle, then generalizes his observations to the rest of the puzzle. This two-tiered approach, reminiscent of professional mathematical techniques, coupled with Bierce’s desire to completely understand the puzzle, suggests a mathematical sophistication hitherto not suspected of him.

Bierce begins his account of the Gem Puzzle with the claim, almost certainly fictitious, that he learned how to solve the puzzle from a “Chinese gentleman” who was familiar with it from his upbringing in China. Anti-Chinese prejudice was very strong in the West in the late nineteenth-century and Bierce was one of the few Western writers—Mark Twain, Bret Harte, and Dan De Quille were three others—who were friendly toward the Chinese and championed them. As the editor of the Argonaut, Bierce more than once wrote against those who persecuted them. But his main expression of sympathy for the Chinese may be found in “The Haunted Valley,” an early short story in the (unusual for Bierce) local color mode, which was published in the Overland Monthly for July 1871 and deals with a socially proscribed love affair between a white man and a Chinese woman. Its local color character and the fact that the story was published in the Overland Monthly both suggest a connection to Bret Harte, who had recently been editor of it. Indeed, Bierce’s jocular quotation of “heathen Chinee” in the third from the last paragraph is a clear reference to Harte’s famous poem “Plain Language from Truthful James” (1870). Bierce’s reference in his Argonaut article to the “Chinese gentleman” was thus probably intended to impute cultural sophistication to the Chinese and thus counter the popular regard of them as wily but uneducated heathens. Except for this paragraph, the rest of the article focuses on the puzzle.

As Bierce moves into his analysis, he assumes that the top three rows of the puzzle are in order, which is easily done by anyone familiar with the puzzle. If one is allowed to pick up the last three blocks numbered 13, 14, and 15 and lay them down in any order, then there are six ways the blocks can be arranged on the bottom line, assuming that the open slot is on the far right. Bierce calls these reorderings derangements of the three numbered blocks and separates them into two categories, reducible and irreducible (see Figure 2).

Bierce claims that the reducible possibilities are those from which one can slide blocks around to get to the solution; the irreducible possibilities are those from which it seems impossible to get to the solution.
To this point Bierce’s analysis is similar to many others that were in print at the time. However, in the second half of the article, Bierce adds another layer of complexity to his analysis. He implicitly considers derangements of three consecutive blocks that appear not only in rows other than the last one but in columns as well. His subsequent analysis is somewhat convoluted but the gist of it is this: when you try to put an irreducible configuration of blocks into order by sliding them around, you will necessarily introduce another irreducible configuration somewhere else in the puzzle. Bierce uses this observation to formulate a second table, where he categorizes configurations of the fifteen blocks that are solvable by sliding in terms of reducible and irreducible configurations. This characterization describes very general conditions under which the puzzle may be solved.

Bierce’s categorization is not mathematically rigorous and his article contains occasional small errors, oversights, and oversimplifications. Nonetheless, it shows that Bierce came close to completely understanding the Gem Puzzle. To better understand exactly what Bierce did, it helps to introduce a bit of notation that Bierce did not use. The process of exchanging the two touching blocks in the puzzle is called a transposition of the blocks. This is an operation that differs from sliding blocks around; physically, it corresponds to lifting two adjacent blocks out of the puzzle, then putting them back into the puzzle in each other’s places. As a result, there is no reason to believe that when two touching blocks are transposed the resulting puzzle can be solved by sliding. For example, starting with the blocks 13, 14, and 15 in order in the bottom row (the solution), if blocks 13 and 14 are transposed, the new order of the blocks is 14, 13, then 15. This is the Class B derangement 14 13 15 from Figure 2, and Bierce claims that a puzzle with blocks in this order cannot be solved by sliding.

We now restate Bierce’s results in terms of transpositions. First, by carefully picking which blocks in the bottom row to transpose and repeating the process multiple times, it is possible to start with the solution configuration and generate every derangement in Figure 2. In addition, it is a fact that one can transform the solution configuration to any Class A derangement using
an even number of transpositions, and to any Class B derangement using an odd number of transpositions. Therefore, Bierce’s observations in Figure 2 can be paraphrased as “the only derangements of the bottom row that can be solved by sliding are those that can be transformed to the solution using an even number of transpositions.” Similarly, Bierce’s general characterization also has a description in terms of transpositions. Start with any arbitrary ordering of the fifteen blocks. If the ordering can be transformed into the solution configuration by an even number of transpositions, then it is possible to solve the puzzle by sliding blocks around. On the other hand, if the ordering differs from the solution by an odd number of transpositions, no solution by sliding is possible. In fact, this description completely describes what actually can and cannot be done with the Gem Puzzle.

Technically speaking, Bierce’s argument has two shortcomings, both related to the mathematical requirement of rigorous proof. First, he claims that every possible configuration of fifteen blocks can be built in stages from reducible and irreducible derangements. However, in Bierce’s last table, where he lists configurations which are solvable and which are not, he doesn’t cover all possible combinations of reducible and irreducible configurations. As a result, although Bierce later claims that his characterization includes all possible puzzle configurations, he doesn’t prove it. Second, and more important, he needs to prove that his two categories have no configuration of the fifteen blocks in common.11 In particular, Bierce claims that if a configuration of fifteen blocks can be solved by sliding, the number of transpositions needed to reduce the configuration to the solution is always an even number, no matter how it is done. This second shortcoming is quite subtle, and forms a significant part of a subsequent formal mathematical analysis. In the end, although both claims are correct, mathematically they would need to be proved, but the proofs are complicated and Bierce does not work them out.

Not only is Bierce’s conclusion interesting, but so is his line of reasoning. In particular, a technique he applies from formal logic deserves mention. In the latter half of his argument he states:

Up to this point it has only been shown that none of the permutations of one class\textsuperscript{12} can be converted into a permutation of the other class, without disturbing the order of some of the other numbers. That is to say, that by such disturbance the one set of permutations may be converted into the other.

The implication in the first sentence is rewritten into the logically equivalent second sentence via the contrapositive.\textsuperscript{13} It is notoriously easy to make a mistake when rewriting statements in an equivalent logical form. Bierce not
only uses the contrapositive form correctly, but builds on it in a significant way in his analysis.

As interest in the Gem Puzzle peaked, mathematically rigorous analyses of the puzzle began to appear. Hermann Schubert put together a mathematically precise argument that gave one of the first impossibility proofs of the swapped configuration on the right side of Figure 1 in the 6 April 1880 issue of the *Hamburgischer Correspondent* (15 Puzzle 117–19). The *American Journal of Mathematics* published two proofs concerning the Fifteen Puzzle by mathematicians William Johnson and William Story in mid-April 1880. It is unlikely that Bierce saw any of these references before he wrote his column, for not only were the references difficult to come by, but he also used different notation and reasoning in his writing. However, it is worth noting that Bierce’s analysis contained some of the arguments used by both Schubert and Story. In the end, Bierce’s article is an impressive accomplishment, a solid and unusually good amateur attempt at a solution to the Gem Puzzle.

The thoroughness that Bierce displayed in devising a solution to the puzzle is a manifestation of his devotion to Reason, which Bierce aspired to follow in his life and his works. In its philosophical forms, both theoretical and practical, Reason underlies most of the social and political commentaries he wrote for his newspaper columns and it is a main theme in most of his fiction. He assigned a tragic significance to Reason in his stories, treating it as humankind’s most important—but still inadequate—weapon in the ultimately deadly battle of existence. In those stories, Reason always functions in a practical, real-life, and therefore complicating context. Probably because of the subject matter on which it is employed, Reason’s use in Bierce’s essays is often overlooked, underestimated, or misconstrued as mere political or social partisanship. Indeed, too little attention has been paid in the analyses of his works to the fact that they often have a basis in formal logic and that their conclusions are usually compelled by the logic of the structure of Bierce’s arguments. The mathematical “Gem Puzzle” allowed him for once, however, to display his reasoning abilities on a purely intellectual problem, free from any distracting considerations of real life and its practical consequences. It was a one-time, never-repeated opportunity for him to operate on an emotionally neutral topic, to use Reason alone to solve a narrowly defined problem without social, political, or moral coloration.

Less than a decade afterwards, Bierce would enter upon his literary high period. By then, the impressive reasoning talents that are revealed in his mathematically logical analysis of “The Gem Puzzle” would be further developed and put to creative and challenging use in devising the intellectual
and morally agonizing puzzles of the brilliant and complex fiction that he composed.

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Notes

2. For background on Bierce’s early years, see Chapter 1 in Carey McWilliams, Ambrose Bierce: A Biography (New York: Boni, 1929).
3. The creation of the Fifteen Puzzle is often attributed to the great American puzzlist Sam Loyd, who himself made that claim. But the Fifteen Puzzle was actually invented by a New York postmaster, Noyes P. Chapman. See Jerry Slocum and Dic Sonneveld, The 15 Puzzle Book (Beverly Hills: Slocum Puzzle Foundation, 2006), pp. 98–109. Subsequent citations are indicated parenthetically.
5. An overview of the responses to the puzzle is offered in 15 Puzzle, pp. 26–40. Bierce’s article is not among those reported.
6. It helped that Bierce had the space of an entire article to devote to the topic, but even taking this into consideration his analysis is concise and unusually perceptive.
7. See, for example, Argonaut, 9 March 1878, p. 9.
8. Bierce does not include the solution configuration as a reducible ordering, since the solution is technically not a derangement of the three numbered blocks.
9. The word “seems” is important here. Simply because someone has been unable to solve the puzzle from these configurations does not mean that the task is impossible. Perhaps the person has just not yet determined the appropriate series of moves. For a mathematically rigorous proof, one would have to show that it is impossible to reach the solution no matter what series of moves were made.
10. To be precise, the two blocks must share an edge. Two blocks that meet only at one of their corners cannot be transposed.
11. In mathematics, this condition is known as being “well-defined.”
12. The term “class” here refers to the Class A and B derangements in Figure 2. Bierce is using the terms “derangement” and “permutation” interchangeably.
13. A conditional statement is of the form “If A, then B.” The contrapositive form of this statement is “If not B, then not A.” Both forms are logically equivalent. Therefore, if the conditional statement is valid, so is its contrapositive. As an explicit example, take the conditional statement “If it is raining, then the ground gets wet.” The contrapositive form of this statement is “If the ground does not get wet, then it is not raining.”

The Gem Puzzle

In conversation, a few days since, with a Chinese gentleman—there is more than one of the race in this city fully entitled to the appellation—speaking of the manners, habits, and customs of his countrymen, he incidentally mentioned that they had in China a
great variety of puzzles, of which comparatively few had ever gone abroad. They formed, he said, the staple of amusement for people of leisure, and were sometimes composed of costly material and were of rare workmanship. This led me to ask him if he had ever seen in China what has been called here the “Gem Puzzle.” He said it was one of the simplest and most common, to be found in every house; it served to amuse the children and the inexpert. He seemed to wonder greatly at the popular interest it had excited in the United States. Especially was he astonished when he heard that men of great intelligence—mathematicians and other savants—had differed, and almost quarreled, upon the subject. He said: “The whole thing is so simple that I think I can show you in a few moments what derangements of the fifteen numbers can and cannot be reduced to order.” At his request I sent out and procured a box of sixteen pieces. My visitor then furnished the following analysis:

The puzzle consists in reducing to numerical order the deranged numbers, the operator being confined to the limited movement of filling a vacant space with a contiguous number. The vacant square may be shifted to any and every square in the box. The operator is in each movement limited to one of three moves. No matter what the derangement may be, a very little practice will enable the operator to bring into numerical order the first three first rows of figures. This will necessarily leave the fourth or bottom row with the three last numbers in some one of its permutations. Now, three figures are susceptible of six permutations. Thus the three figures comprising the last line, or bottom row—all the preceding numbers being reduced to numerical order—may be left in either of the six forms following:

Now, it is to be observed that there is a feature belonging to the third and fifth derangements that does not attach to the other three. It is this: that in these two the numerical order can be restored by transferring, in the third, the 13 on the extreme right to the extreme left; and, in the fifth, by transposing the 15 on the extreme left to the extreme right—that is, by transposing the extremes. Whereas, in the second, fourth, and sixth derangements, numerical order can only be restored by interposing some one of the numbers between the other two. The change of extremes can be effected without disordering, or inverting, any of the numbers on the first, second, or third lines; the interposition cannot be made without an inversion of some of the preceding numbers. That is to say, the limited sliding movement to which the operator is confined permits the first character of transfer, and not the second. Let us now call the two derangements which can be reduced to numerical order by transfer of their extremes class “A,” and the three that are wanting in this characteristic, class “B.” They may be presented to the eye by the following diagram:

As a practical illustration of the possibility of the reduction of the one class and not

<table>
<thead>
<tr>
<th>First</th>
<th>Second</th>
<th>Third</th>
<th>Fourth</th>
<th>Fifth</th>
<th>Sixth</th>
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</thead>
<tbody>
<tr>
<td>13 14 15</td>
<td>13 15 14</td>
<td>14 15 13</td>
<td>14 13 15</td>
<td>15 13 14</td>
<td>15 14 13</td>
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</tbody>
</table>

Class “A.” (Reducible)
13 15 14
15 13 14

Class “B.” (Irreducible)
14 15 13
14 13 15
15 14 13
of the other, arrange the three first lines in numerical order and leave the last line in either derangement of the reducible class, say 14, 15, 13.

Move the 12 into the space left vacant by the removal of the 16—9, 10, 11 right—14 up—15, 13, 12 left—11 down—14, 9, 10 right—15 up.

Up to this time no change has been made in the order of the numbers. If we should reverse the order of our movements, we should be just where we started. But the condition now is such that the three disordered numbers are contiguous to the vacant space through which we can move them ad libitum. It will be seen at a glance that the sliding movement to which the operator is confined will permit the change of the extremes, but will not permit the interjection of either one number between the other two. And here we should note that because transfer of extremes is possible, any one of either class may, by the sliding movement, be converted into any one of the same class, so that class “B” being inconvertible to class “A,” and class “A” being convertible into the sixth permutation, which is the numerical order, it follows that class “B” can not be converted into that permutation which presents the numerical order.

Hence we conclude that the three first lines being arranged in numerical order, the fourth or last line, disordered in either of the two forms constituting class “A,” the whole may be reduced to numerical order. On the other hand, if the last line is left in either of the three forms of class “B,” the reduction is impossible.

But there is something more to be considered in this connection. Up to this point it has only been shown that none of the permutations of one class can be converted into a permutation of the other class, without disturbing the order of some of the other numbers. That is to say, that by such disturbance the one set of permutations may be converted into the other. If then, instead of one, we should have two “B” derangements, it follows that the same operation that converts the first “B” into an “A,” disturbing, as it necessarily does, the second “B,” that is converting it into an “A,” the two “B’s” will be converted into two “A’s.” On the other hand, if it is at the expense of an “A” derangement we seek to cure the “B” derangement, the result is that, the one being converted into the other, we are left with an irreclaimable “B.” From these premises we may formulate the following tables:

**Practicable.**

1. One or more “A” derangements.
2. An “A” derangement with any even number of “B” derangements.
3. Any even number of “B” derangements, which leaves us for the

**Impracticable.**

1. An “A” derangement with any odd number of “B” derangements.
2. Any odd number of “B” derangements.

Nobody, says my Chinese friend, ever has or ever will solve this puzzle under either of the conditions herein designated as “impracticable” [sic]: while with the conditions pronounced “practicable,” the expert will dispose of the case in the time required to move the pieces—and on these propositions he says “You may bet your bottom dollar.”

If this disquisition of the “heathen Chinee” serves no other purpose, it will enable the reader to test the ingenuity of the inexpert members of his household in reducing to order a variety of “practicable” derangements without puzzling his brain over an “impracticable” puzzle.

There is one other matter connected with this curious puzzle to which my attention
was called, and which I learned had been the subject of grave discussion in the American papers. The problem we have been considering is to reduce the deranged numbers, not only to numerical order, but to the specific numerical order in which the lines shall run from left to right. Now, let us assume that, having reduced the three first lines to order, we are left with a derangement of class “B,” say the 13, 15, 14, in the fourth line. Move these three numbers to the right, so that the 14 will fill the space left vacant by the original removal of the 16; draw down the 1, 5, 9; 2, 3, 4 left; 8, 12, 14 up; 9, 15, 13 right, and so proceed until the first line to the left reads 1, 2, 3, 4. Then the 15, 13, 14 will occupy the fourth line to the right. Now, reckoning the rows up and down, instead of from left to right, you will have the impracticable 13, 15, 14, together with another “B” derangement; in other words, you will have an even, instead of an odd, number of “B” derangements, which is found in our table under the “practicable” head.

Again, if in reducing the derangement or derangements to order, reckoning from left to right, the fourth line assumes either the numerical order, or either of the “A” derangements, and, by the operation just referred to, the 1, 2, 3, 4 are caused to occupy the first row to the left, you will then have the bottom line transferred to the right-hand row. But now the “A” derangement will be accompanied with one “B” derangement, which brings it under the “impracticable” head. Hence we find that the solution that is impracticable, reckoning the numerical order from left to right, becomes practicable reckoning up instead of across, and vice versa. Therefore, if the terms of the problem be so changed as to include either mode of counting, there is no possible derangement of the numbers that cannot be reduced to numerical order.

SAN FRANCISCO. May 10, 1880.